TWO APPROACHES TO ONE OPTIMIZATION PROBLEM IN RECOGNITION THEORY

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Abstract: *In recognition theory a number of optimization problems appear. The search for the maximum solvable subsystem of the system of linear inequalities is one of these tasks. In this paper two approximation solution methods for given problem are proposed. The first method is based on the theory of nodal solutions of systems of linear inequalities. But the second algorithm principally differs from the first: it is based on some reasoning of geometric nature. The comparison of the represented methods is produced in the number of problems.*

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Introduction

The solution of optimization problems in recognition theory is one of the most important stages in the synthesis of the high-precision algorithms of classification, recognition and forecast. The following optimization problems are here central: search for the maximum solvable subsystems of systems of inequalities, search for the minimum coverings of matrices, synthesis of the minimum formulas for realization of weakly determined Boolean and finite-valued functions, constructing the algebraic correctors of minimum degree, and others.

During the optimization of some models of recognition algorithms, e.g., the estimate-calculating algorithms (ECAs), a specific system of conditions is constructed. This system is described by the large number of linear inequalities and, as a whole, can be contradictory. It is necessary satisfy these conditions as precisely as possible. In the general case, the solution of this problem is reduced to the search for the maximum (according to the number of inequalities) solvable subsystem of the assigned system of linear inequalities.

In some cases, additional requirements are imposed on the desired solvable subsystem. For example, the given system of linear inequalities may be broken into blocks (subsystems) of identical power. It is necessary to find a solvable subsystem that contains the maximum number of complete blocks.

In other cases, we need to find the maximum solvable subsystem that contains the fixed inequalities of the assigned system. This task appears during the dynamic completion of the training sample.

Formulation of the problem

In our paper, we consider the general task of the search for the maximum solvable subsystem (MSS) of the system of linear inequalities. Assume that an inconsistent system of linear inequalities in the following form (1) is assigned:

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i, \quad i = 1,\dots,m. \tag{1}$$

It is necessary to find the MSS of this system.

One approach to the solution of this problem is known (see [Chernikov, 1968], chapter 5). This method uses the convolution of the system of linear inequalities. However, it is unfit for the tasks of large dimensionality, since it requires the rapidly growing memory capacity.

Earlier the author constructed the precise solution algorithm for this problem (see [Katerinochkina, 2005]). This algorithm is effective for linear systems of small ranks with a large number of inequalities. Furthermore, the modifications of general method for some types of tasks with additional requirements on the desired solvable subsystem were developed. However, the precise algorithms of solution of such problems make large exhaustive search. Therefore, the rapid approximation methods should be developed for the tasks of large dimensionality.

In this paper, the author proposed two approaches to the solution of the formulated problem. The first approach is based on examining a special set of subsystems of the assigned system of linear inequalities. The second method uses some reasoning of geometric nature. Descriptions of both methods are given, and the comparison of their work in the number of tasks also is produced.

Description of the first method

Suppose that an unsolvable system S of form (1) is given.

DEFINITION 1. We say that a solvable subsystem P of system S is a *not extensible, solvable subsystem*, if the addition to P of any (not entering it) inequality of system S converts it into an inconsistent system.

The following assertion is valid.

LEMMA 1. Assume that the inconsistent system S has a rank $r, r > 0$. Then the rank of any not extensible, solvable subsystem of system S is equal to r.

DEFINITION 2. A subsystem whose power and rank are equal to r is called *an r-subsystem.*

Following [Chernikov, 1968], let us introduce a number of concepts necessary for further reasoning. Let us examine system P of form (1). We assume that system P is solvable and has a rank r different from zero.

DEFINITION 3. We call the solution of system P *a nodal solution*, if it turns into the equalities some r of its inequalities with linearly independent left sides.

DEFINITION 4. The subsystem of system P is called *a nodal subsystem*, if its rank is equal to the number of inequalities in it and all of its nodal solutions satisfy system P.

In [Chernikov, 1968] the following theorem is proven.

THEOREM 1.1. Each solvable system of the linear inequalities of form (1) with rank $r, r > 0$, has at least one nodal subsystem, and, hence, at least one nodal solution (each nodal subsystem is an r-subsystem).

Examine now an inconsistent system S in the form (1). Lemma 2 follows from Lemma 1 and Theorem 1.1.

LEMMA 2. Let the inconsistent system S have the rank r, $r > 0$. Then each not extensible, solvable subsystem of system S has at least one nodal r - subsystem.

Therefore, for the selection of all not extensible, solvable subsystems of system S, it is sufficient to examine all of its r -subsystems. For each of them, it is required to solve a simple task, i.e., to find one nodal solution. For this purpose, it is necessary to replace all inequality signs in a subsystem with the equalities signs and find one solution of the obtained system of linear equations. The obtained nodal solution must be substituted into all inequalities of system S, and the inequalities satisfied with this solution must be selected. The chosen inequalities form a solvable subsystem. After exhaustion of r -subsystems, we will obtain a certain set that contains all not extensible, solvable subsystems of system S. Among them, it is possible to select optimum subsystems with the required properties. Maximum subsystem (according to the power) among the chosen subsystems is the MSS of system S.

The described method combines finding the MSS with the simultaneous presence of one solution, which is the final goal of optimization problem.

However, the complete exhaustion of r -subsystems will be too long for tasks of large dimensionality. Therefore, a number of approximations methods have been developed.

In this paper, the approximate algorithm **A$_w$** of solution of the formulated problem is presented. This algorithm realizes a partial directed search of r -subsystems of the assigned system.

General work scheme of algorithm A$_w$. First, we transform system S of form (1) as follows. We normalize the inequalities of the system so that the modulus of the first nonzero coefficient in each inequality would be equal to 1 (thus, we do not change the signs). Then, we place all inequalities in order of increasing of right sides and, afterwards, we examine inequalities in this order.

Algorithm **A$_w$** has several iterations. At any iteration, we first choose the so-called "base r -subsystem". Then, a set of associated subsystems is constructed by replacing some inequalities in the base r -subsystem. We apply the construction procedure of *the extended solvable subsystem* (ESS) to the base subsystem and associated r - subsystems; at the same time, we find one solution for each ESS. The maximum subsystem is selected from the obtained ESSs. The power of the latter subsystem is compared with value ρ, *i.e.,* the record power of the ESS obtained at previous iterations.

As a result, after all iterations, the record value of power ρ and the corresponding solution $\mathbf{x}$ are remembered. Maximum ESS, which is considered to be an approximation of the MSS of system S, is restored by this solution.

Let us now describe the procedures mentioned in the general work scheme of the algorithm.

Construction of base r –subsystems. Algorithm **A$_w$** has a parameter w that determines the maximum power of the intersection of base subsystems. If $r = 1$, we set $w = 0$; if $r = 2$, then $w = 1$. Generally, at $r > 2$, values of parameter w fall in following boundaries: $0 \le w \le r - 3$.

The first r (in the indicated order) inequalities with the independent left sides enter into the first base subsystem. For constructing the next base subsystem, we move the first $r - w$ inequalities, which entered into the previous base subsystem, away from system S. We obtain a reduced system of linear inequalities, from which we again select the first r independent inequalities. We continue this process until it is possible to choose an r-subsystem from the reduced system. The total number b of base subsystems is estimated as follows:

$$b \le \left[\frac{m-r}{r-w} \right] + 1 .$$

Construction of the extended solvable subsystem (ESS). This procedure can be applied to any r-subsystem of system S. Let the r-subsystem B be given. First, we find the nodal solution of B. For this, we replace all inequality signs in the subsystem by the equality signs and find the solution of the obtained system of linear equations. We find only one solution $\mathbf{x}$; i.e., at $r < n$, we select one solution from the set. We then substitute the obtained solution $\mathbf{x}$ in all inequalities of system S. Also, we separate the inequalities satisfied by $\mathbf{x}$; these inequalities form the ESS. We remember the power ρ of the obtained ESS and the corresponding solution $\mathbf{x}$.

Construction of the set of subsystems associated with the base subsystem. The construction of associated subsystems for the base subsystem is tree-like process. Assume that, on some iteration, a base subsystem B is built. Let us apply the procedure for constructing an ESS. Thus, its nodal solution $\mathbf{x}$ is found and subsystem $R(B)$ (extension of B) with power ρ is built. Let us select the inequalities from system S that $\mathbf{x}$ does not satisfy. The set of these inequalities is denoted by $\overline{R}$.

STAGE 1. First we construct subsystems associated with the base r-subsystem B. These subsystems are subsystems of the first level. For this, we replace each of the r inequalities of system B with each of the $m-\rho$ inequalities from the set $\overline{R}$ in turn. We will obtain $r(m-\rho)$ subsystems associated with B; of these only r-subsystems are left. To each received r-subsystem B_i, we shall apply the procedure for construction the ESS. As a result, the extended solvable subsystem $R(B_i)$ with power ρ_i will be found.

DEFINITION 5. We call an r-subsystem B_i, which is associated with the base subsystem B, *the point of increase*, if $\rho_i > \rho$.

STAGE 2. Following cases are possible:

1) Among subsystems associated with B, there are no points of increase. Then, this branch of the process is broken and the transition to the next base subsystem is produced.

2) Points of increase exist and, among them, there is one maximum; i.e., for some r-subsystem B_t, we have $\rho_t > \rho$, and $\rho_t > \rho_i$ for all $i \ne t$. Then, for subsystem B_t, the associated subsystems are constructed exactly as was done for base subsystem B, which are subsystems of second level.

3) There are several points of the maximum increase, including subsystems $B_{j_1},...,B_{j_l}$, for which will be $\rho_{j_1},...,\rho_{j_l}$. In this case, we investigate these subsystems one by one until we find the point of increase for one of them.

Then we pass to the third stage, namely, we construct, as are above, the subsystems of third level for the maximum point of increase. We do not return to the remaining subsystems of the second level.

Let us put the following limitation on number u of the levels of this process:

$$u \le \left]\frac{r-w}{2}\right[-1 .$$

This is the result of the requirement that the associated subsystems for the neighboring base subsystems not coincide.

Estimation of the algorithm complexity. After the end of all iterations of algorithm $\mathbf{A}_\mathbf{w}$, we will obtain the record value ρ for the power of all constructed ESSs and the corresponding solution $\mathbf{x}$. The maximum ESS is restored based on this solution. This ESS is considered to be the approximation for MSS of system S.

Let us estimate the number q of all considered subsystems of power r. For one examined r-subsystem B, $q(B)$ subsystems are investigated, where

$$q(B) \le r^2 (m-\rho)^2 u .$$

Further, we have:

$$q \le q(B)b \le r^2 (m-r)^2 bu \le \frac{r^2 (m-r)^3}{2} ,$$

where value b is the number of base subsystems.

This estimation is overstated; the number of the subsystems examined will be much lower in practice.

Description of the second method

This method is principally different from those mentioned above. It is based on some reasoning of geometric nature.

Description of the overall algorithm scheme. For the brevity we will sometimes write system S in the following form:

$$l_i(\mathbf{x}) \le b_i, \quad i = 1,...,m, \tag{2}$$

where $l_i(\mathbf{x}) = (\mathbf{a}_i, \mathbf{x}) \equiv \sum_{j=1}^{n} a_{ij} x_j$.

DEFINITION 5. The inequality $(\mathbf{a}, \mathbf{x}) \le b$ is called *improper*, if vector $\mathbf{a}$ is equal to zero.

Depending on the sign of value b, improper inequality is either contradictory or identical.

It is obvious that the improper inequalities can be not considered in the process of construction of the MSS of a system. Contradictory inequalities will enter not into one subsystem, but identical inequalities can be added to any solvable subsystem.

The following assertion is known (for example, see [Eremin, 2007]).

ASSERTION 1. Any inconsistent system of linear inequalities that does not contain improper inequalities can be subdivided into two solvable subsystems.

Actually, let us divide system (2) into two subsystems as follows.

Denote by M_+ the set of those indices $i \in \{1,\dots,m\}$, for which the first of the nontrivial coefficients $a_{i1},\dots,a_{in}$ is positive. Put also $M_- = \{1,\dots,m\} \setminus M_+$. Then system (2) is decomposed into two subsystems:

$$l_i(\mathbf{x}) \le b_i, \quad i \in M_+; \tag{2.1}$$

$$l_i(\mathbf{x}) \le b_i, \quad i \in M_-. \tag{2.2}$$

It is easy to show that these subsystems are solvable and to find their particular solutions.

Let us describe the work scheme of the search algorithm for the MSS of system S in form (2).

STAGE 0. First we divide the assigned system into two solvable subsystems (2.1) and (2.2).

For each of these subsystems we find one particular solution. Let us denote them by $\mathbf{y}$ and $\mathbf{z}$ respectively.

Then cycle on i, $i = 1,\dots,m$, is produced. In this cycle the i-th iteration consists of several stages.

STAGE i1. Let us note that the hyperplane $l_i(\mathbf{x}) = b_i$ corresponds to the inequality $l_i(\mathbf{x}) \le b_i$ of the given system. First we construct the points $p(\mathbf{y}, i)$ and $p(\mathbf{z}, i)$ that are the projections of points $\mathbf{y}$ and $\mathbf{z}$ on the hyperplane $l_i(\mathbf{x}) = b_i$.

STAGE i2. For the straight line passing through points $p(\mathbf{y}, i)$ and $p(\mathbf{z}, i)$, we find the points of its intersection with all remaining hyperplanes: $l_j(\mathbf{x}) = b_j$, $j = 1,\dots,m,\ j \ne i$. Let these be the points $\mathbf{u}^1,\dots,\mathbf{u}^l$, $(l \le m-1)$.

Stage i3. Then we substitute each obtained point $\mathbf{u}^j, 1 \le j \le l,$ in all inequalities of system (2) and calculate the number of inequalities that this point satisfies. These inequalities obviously form the solvable subsystem. We compare the power of this subsystem with the record power obtained at the previous steps. In this case, we memorize the value of the record power ρ of solvable subsystem and the corresponding point $\mathbf{v}$.

As a result after all iterations, the record value of power ρ and the corresponding solution $\mathbf{v}$ are memorized. On this solution the solvable subsystem is restored. This subsystem is considered as the approximation for the MSS of system S.

Let us estimate the exponent of operations of the described algorithm. Algorithm contains m iterations, for each of them we find not more than m points. We substitute each point in m inequalities from n variables. In all, we obtain the order $m^3 n$ operations.

About the parameters of the represented algorithm. In the initial stage, system (2) is divided into two solvable subsystems determined above. For each of these subsystems it is necessary to find one particular solution. It is obvious that the particular solutions of subsystems (2.1) and (2.2) are built not uniquely.

Consider the coefficients matrix A_+ of system (2.1). By the transposition of its lines it is possible to lead it to the stepped form. In the obtained matrix the first nontrivial column of each step consists of the positive elements. Let the reduced matrix have k of steps. Let us denote by $i_j,\ j=1,\dots,k,$ the number of the first nontrivial column of the j -th step and by l_j denote the number of the upper row of the j -th step. We will search for the particular solution of the subsystem, beginning from the inequalities that correspond to the k -th step of the given matrix.

Obviously, we can attach any values to variables $x_j,\ i_k < j \le n$. Then the subsystem, which corresponds to the k -th step, is led to the form:

$$a_{ji_k} x_{i_k} \le c_j,\ j = l_k, l_k+1,\dots,m, \quad a_{ji_k} > 0.$$

Then we can choose the value of variable x_{i_k} from the condition:

$$x_{i_k} \le \min_{l_k \le j \le m} \frac{c_j}{a_{ji_k}}.$$

Substituting the obtained values of variables into the previous inequalities, we find the values of remaining variables by analogy.

The particular solution of subsystem (2.2) is built according to the same scheme. Difference lies in the fact that the set of the values of the corresponding variable will be limited not on top but from below.

Thus, in the process of the search for the particular solutions we can attach any values to certain (free) variables and to others attach arbitrary values from the infinite set.

Hence such variables can be considered as the parameters. Varying these parameters, we will obtain different particular solutions.

It is possible to use this fact in the process of applying the represented method. Changing the relationships between the parameters, we can attempt to improve initial result. Thus, we can tune our algorithm to the task in the dialogue regime.

Comparison of two solution methods of the considered problem. We made a comparison of the represented methods (method 1 and method 2). About five ten tasks were solved by these methods. The identical results are obtained for 48% of these tasks. For 24% of tasks, method 1 gave the best result. For 28% of tasks, method 2 operated better.

Conclusion

In this paper, two search methods for the maximum solvable subsystem of the system of linear inequalities are proposed. The represented algorithms can be used together with others methods. Let us note that each of them can operate better than other on some tasks.

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