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ON THE EXTERIOR PENALTY FUNCTION METHOD FOR THE CONSTRAINED OPTIMAL CONTROL PROBLEM FOR QUASILINEAR PARABOLIC EQUATIONS

Mahmoud Farag

Abstract: This paper presents the numerical solution of a constrained optimal control problem (COCP) for quasilinear parabolic equations. The COCP Is converted to unconstrained optimization problem (UOCP) by applying the exterior penalty function method. The numerical algorithm for solving UOCP using the conjugate gradient method (CGM) is given. The computing optimal controls are helped to identify the unknown coefficients of the quasilinear parabolic equation. Numerical results are reported.

Keywords: optimal control, quasilinear parabolic equations, penalty function method, finite difference methods.

ACM Classification Keywords: G.1 Numerical Analysis, G.1.6 Optimization, G.1.8 Partial Differential Equations.

Nomenclature and Notations

E_N	The N-dimensional Euclidean space
D	a bounded domain of E_N
Ω	$\{(x,t): x \in D, \ t \in (0,T)\}$
$\phi(x) \in L_2(D), g_0(t), g_1(t) \in L_2(0,T), f_0(t), f_1(t) \in L_2(0,T)$	Given functions
$r_1, r_2, l, T, \omega = (w_1, w_2, \dots, w_N) \in E_N$	Given numbers
$\alpha \geq 0, \beta_0 \geq 0, \beta_1 \geq 0, \beta_0 + \beta \geq 0$ $\zeta_0, \mu_0, \zeta_1, \mu_1 > 0, R > 0$	Given numbers
$U = \{u : (u_1, u_2, \dots, u_N) \in E_N, \|u\|_{E_N} \leq R\}$	Control space
C	Constant not depending on δu
A_m	Positive numbers, $\lim_{m\to\infty} A_m = +\infty$

$L_2(D)$	Banach space which consisting of all measurable functions on D with the norm $\|y\|_{L_2(D)} = \sqrt{\int_D \|y\|^2 \, dx}$
$L_2(\Omega)$	Hilbert space which consisting of all measurable functions on Ω with $\langle y_1, y_2 \rangle_{L_2(\Omega)} = \int_0^l \int_0^T y_1(x,t) \; y_2(x,t) \; dx \; d$ $\|y\|_{L_2(\Omega)} = \sqrt{\langle y, y \rangle_{L_2(\Omega)}}$
$L_2(0,l)$	Hilbert space which consisting of all measurable functions on $(0,l)$ with $\langle y_1, y_2 \rangle_{L_2(0,l)} = \int_0^l y_1(x) \; y_2(x) \; dx \;\; , \;\; \|y\|_{L_2(0,l)} = \sqrt{\langle y, y \rangle_{L_2(0,l)}}$
$W_2^{1,0}(\Omega)$	$W_2^{1,0}(\Omega) = \left\{ y \in L_2(\Omega) \; and \; \frac{\partial y}{\partial x} \in L_2(\Omega) \right\}$ is a Hilbert space with $\langle y, y \rangle_{W_2^{1,0}(\Omega)} = \sqrt{\|y\|^2_{L_2(\Omega)} + \left\| \frac{\partial y}{\partial x} \right\|^2_{L_2(\Omega)}}$ $\|y\|_{W_2^{1,0}(\Omega)} = \int_\Omega [\, y_1 \; y_2 + \frac{\partial y_1}{\partial x} \frac{\partial y_2}{\partial x}] \; dx \; dt$
$W_2^{1,1}(\Omega)$	$W_2^{1,1}(\Omega) = \left\{ y \in L_2(\Omega) \; and \; \frac{\partial y}{\partial x}, \frac{\partial y}{\partial t} \in L_2(\Omega) \right\}$ is a Hilbert space with $\langle y, y \rangle_{W_2^{1,1}(\Omega)} = \sqrt{\|y\|^2_{L_2(\Omega)} + \left\| \frac{\partial y}{\partial x} \right\|^2_{L_2(\Omega)} + \left\| \frac{\partial y}{\partial t} \right\|^2_{L_2(\Omega)}}$ $\|y\|_{W_2^{1,1}(\Omega)} = \int_\Omega [\, y_1 \; y_2 + \frac{\partial y_1}{\partial x} \frac{\partial y_2}{\partial x} + \frac{\partial y_1}{\partial t} \frac{\partial y_2}{\partial t}] \; dx \; dt$
$V_2^{1,0}(\Omega)$	Banach space consisting of elements the space $W_2^{1,0}(\Omega)$ with the norm $\|y\|_{V_2(\Omega)} = vrai \max_{0 \le t \le T} \|y(x,t)\|_{L_2(D)} + \sqrt{\int_\Omega \left\| \frac{\partial y}{\partial t} \right\|^2 dx}$
$V_2^{1,0}(\Omega)$	Subspace of $V_2(\Omega)$, the elements of which have in sections $D_t = \{(x,\tau) : x \in D, \tau = T\}$ traces from all $t \in [0,T]$, continuously charging from $t \in [0,T]$ in the norm $L_2(D)$.

Introduction

Optimal control problems for partial differential equations are currently of much interest. A large amount of the theoretical concept which governed by quasilinear parabolic equations has been investigated in the field of optimal control problems [Belmiloudi, 2004], [Ryu, 2004]. These problems have dealt with the processes of hydro- and gasdynamics, heat physics, filtration, the physics of plasma and others [Farag, 2004], [Iskenderov, 1974]. From the mathematical point of view, the definition and refinement of the unknown parameters of the model present the problem of identification and optimal control of partial differential equations. The importance of investigating the identification and optimal control problems was developed in [Lions, 1973], [Farag, 2003].

This paper presents the numerical solution of a constrained optimal control problem (COCP) for quasilinear parabolic equations. The COCP Is converted to unconstrained optimization problem (UOCP) by applying the exterior penalty function method [Xing,1994]. The numerical algorithm for solving UOCP using the conjugate gradient method is given [Damean, 2000]. The computing optimal controls are helped to identify the unknown coefficients of the quasilinear parabolic equation. Numerical results are reported.

The heat exchange process described by a partial differential equation of quasilinear parabolic type as follows:

$$\frac{\partial y}{\partial t}-\frac{\partial}{\partial x}\left(\lambda\,(y,u)\,\frac{\partial y}{\partial x}\right)+Z\;(y,u)\;y=F(x,t)\;,(x,t)\in\Omega \tag{1}$$

with the initial condition and the boundary conditions

$$y\,(x,0)=\phi\,(x)\quad,\;x\in D \tag{2}$$

$$\lambda\,(y,u)\left.\frac{\partial y}{\partial x}\right|_{x=0}=g_0(t)\;,\;\lambda\,(y,u)\left.\frac{\partial y}{\partial x}\right|_{x=l}=g_1(t)\,,\,0\le t\le T \tag{3}$$

On the set U under the conditions (1)-(3) and additional restrictions

$$\zeta_0\le\lambda\,(y,u)\le\mu_0\,,\,\zeta_1\le Z(y,u)\le\mu_1\,,\,r_1\le y(x,t)\le r_2 \tag{4}$$

is required to minimize the function

$$J_\alpha(u)=\beta_0\int_0^T\left[y\,(0,t)-f_0(t)\right]^2dt+\beta_1\int_0^T\left[y\,(l,t)-f_1(t)\right]^2\,dt+\alpha\left\|u-\omega\right\|_{E_N}^2 \tag{5}$$

The solution of the reduced problem (1)-(3) explicitly depends on the control u. Therefore, we shall also use the notation $y=y\,(x,t;u)$. Based on adopted assumptions and the results of [Ladyzhenskaya, 1973] follows that for every $u\in U$ the solution of the problem (1)-(4) is existed, unique and $|y_x|\le C,\,\forall\,(x,t)\in\Omega,\,\forall\,u\in U$.

Optimal control problems of the coefficients of differential equations do not always have solution [Goebel, 1979]. In [Farag, 2003], we proved the existence and uniqueness of the solution of problem (1)-(5) as follows:

Lemma (1)

At above adopted assumptions for the solution of the reduced problem (1)-(5) the following estimation is valid

$$\|\delta y\|_{V_2^{1,0}(\Omega)} \le C\sqrt{\left\|\lambda(y,u)\frac{\partial y}{\partial x}\right\|^2_{L_2(\Omega)} + \|\delta Z(y,u)\,y\|^2_{L_2(\Omega)}} \quad (6)$$

Lemma (2): The function $J_0(u)$ is continuous on U.

Theorem (1) : The problem (1)-(5) at any $\alpha \ge 0$ has at least one solution.

Theorem (2): The problem (1)-(5) at any $\alpha > 0$, at almost all $\omega \in E_N$ has a unique solution.

UOCP and CBVP

The inequality-constrained problem (1) through (5) is converted to a problem without inequality constraints by adding a penalty function to the objective (5), yielding the minimizing following function:

$$\Upsilon_{\alpha,m}(u) = f_\alpha(u) + A_m \int_0^l\int_0^T [\, B_1(y,u) + B_2(y,u) + P_1(y) + P_2(y)]\, dx\, dt \quad (7)$$

under the condition (1)-(3), where

$$P_1(y) = [\max\{r_1 - y(x,t;u);0\}]^2 \qquad B_1(y,u) = [\max\{\zeta_0 - \lambda(y,u);0\}]^2 + [\max\{\lambda(y,u) - \mu_0;0\}]^2$$

$$P_2(y) = [\max\{y(x,t;u) - r_2;0\}]^2 \qquad B_2(y,u) = [\max\{\zeta_1 - \lambda(y,u);0\}]^2 + [\max\{\lambda(y,u) - \mu_1;0\}]^2$$

The problem (7) and (1)-(3) is called **UOCP**. It is assumed that the following conditions are fulfilled:

a) The functions satisfy the Lipshitz condition for u.

b) The first derivatives of the functions $\lambda(y,u)$, $Z(y,u)$ with respect to u are continuous functions.

c) For any $u \in U$ s.t. $\|u\|_{E_N} \le R$ the functions $\dfrac{\partial \lambda(y,u)}{\partial u}, \dfrac{\partial Z(y,u)}{\partial u}$ belong to $L_\infty(\Omega)$.

d) $\displaystyle\int_0^l\int_0^T \frac{\partial \lambda(y,u)}{\partial u}\, dx\, dt$, $\displaystyle\int_0^l\int_0^T \frac{\partial Z(y,u)}{\partial u}\, dx\, dt$ are bounded in E_N

Theorem (3): It is assumed that the above conditions are satisfied. The function $\Psi(x,t) \in W_2^{1,1}(\Omega)$ is a solution in of the following conjugate boundary value problem (**CBVP**) [Vassiliev, 1980] :

$$\frac{\partial\Psi}{\partial t}+\frac{\partial}{\partial x}\left(\lambda(y,u)\frac{\partial\Psi}{\partial x}\right)-\frac{\partial\lambda(y,u)}{\partial y}\frac{\partial\Psi}{\partial x}\frac{\partial y}{\partial x}-\left(\frac{\partial Z(y,u)}{\partial y}\,y+Z(y,u)\right)\Psi$$
$$=A_m\left[\frac{\partial B_1(y,u)}{\partial y}+\frac{\partial B_2(y,u)}{\partial y}+\frac{\partial P_1(y)}{\partial y}+\frac{\partial P_2(y)}{\partial y}\right],(x,t)\in\Omega \tag{8}$$

$$\Psi(x,T)=0\ ,\ x\in D \tag{9}$$

$$\lambda(y,u)\frac{\partial\Psi}{\partial x}=\begin{cases}2\beta_0\,[\,y(x,t)\,-f_0(t)\,] & x=0\\ -2\beta_1\,[\,y(x,t)\,-f_1(t)\,] & x=l\end{cases}\quad ,\quad 0\le t\le T \tag{10}$$

where y is the solution of problem (1)-(3) for $u\in U$.

For the sufficient differentiability conditions of the function $\Upsilon_{\alpha,m}(u)$, we have the following theorem:

Theorem (4): It is assumed that the above conditions are satisfied. The function $\Upsilon_{\alpha,m}(u)$ is Frechet differentiable and its gradient satisfies the equality

$$\frac{\partial\Upsilon_{\alpha,m}}{\partial u}=-\frac{\partial H}{\partial u}\equiv\left(-\frac{\partial H}{\partial u_1},-\frac{\partial H}{\partial u_2},\ldots,-\frac{\partial H}{\partial u_N}\right) \tag{11}$$

where $H(y,\Psi,u)$ is the Hamiltonian function defining as follows

$$H\ (y,\Psi,u)\equiv-\int_\Omega[\,\lambda\,(y,u)\,\frac{\partial\Psi}{\partial x}\,\frac{\partial y}{\partial x}+Z(y,u)\;y\;\Psi$$
$$+\;A_m\{B_1(y,u)+B_2(y,u)\}\;]\;dx\;dt\;-\alpha\;\|u-\omega\|_{E_N}^2 \tag{12}$$

The Iterative Algorithm for Solving UOCP

The following iterative algorithm is developed (k being iteration numbers). In view of relations (1)-(7) one considers the following iterative algorithm:

Step 0: Choose $k=1\ ,\ u_{Nc}\in V\ ,\omega_{Nc}\in E_N\ \ ,\ \varepsilon_1>0\ ,\ \varepsilon_2>0$

Step 1: Compute $Y^{(k)}(x,t)$, that is, the state system described by equations (1)-(4):

$$\frac{\partial y^{(k)}}{\partial t}-\frac{\partial}{\partial x}\left(\lambda\,(y^{(k)},u^{(k)})\,\frac{\partial\, y^{(k)}}{\partial\, x}\right)+Z\,(y^{(k)},u^{(k)})\;y^{(k)}=F(x,t)\,,(x,t)\in\Omega$$

$$y^{(k)}(x,0)=\phi\,(x)\;\;,\;x\in D$$

$$\lambda\,(y^{(k)},u^{(k)})\,\frac{\partial\, y^{(k)}}{\partial\, x}\bigg|_{x=0}=g_0\,(t),\;\lambda\,(y^{(k)},u^{(k)})\,\frac{\partial\, y^{(k)}}{\partial\, x}\bigg|_{x=l}=g_1\,(t)\,,\,0\le\,t\,\le T$$

Step 2: Compute $\Psi^{(k)}(x,t)$, that is, the adjoint state system described by equations (10)-(14)

$$\frac{\partial\,\Psi^{(k)}}{\partial\, t}+\frac{\partial}{\partial\, x}\left(\lambda\,(y^{(k)},u^{(k)})\,\frac{\partial\,\Psi^{(k)}}{\partial\, x}\right)-\frac{\partial\lambda\,(y^{(k)},u^{(k)})}{\partial\, y^{(k)}}\,\frac{\partial\,\Psi^{(k)}}{\partial\, x}\,\frac{\partial\, y^{(k)}}{\partial\, x}$$

$$-\left(\frac{\partial Z(y^{(k)},u^{(k)})}{\partial\, y^{(k)}}\;y^{(k)}+Z(y^{(k)},u^{(k)})\right)\Psi^{(k)}$$

$$=A_m\left[\frac{\partial\, B_1(y^{(k)},u^{(k)})}{\partial\, y^{(k)}}+\frac{\partial\, B_2(y^{(k)},u^{(k)})}{\partial\, y^{(k)}}+\frac{\partial\, P_1(y^{(k)})}{\partial\, y^{(k)}}+\frac{\partial\, P_2(y^{(k)})}{\partial\, y^{(k)}}\right],(x,t)\in\Omega$$

$$\Psi^{(k)}(x,T)=0\;,\;x\in D$$

$$\lambda(y^{(k)},u^{(k)})\,\frac{\partial\,\Psi^{(k)}}{\partial\, x}=\begin{cases}2\,\beta_0\;[\;y^{(k)}(x,t)\;-f_0(t)\;] & x=0\\ -2\,\beta_1\;[\;y^{(k)}(x,t)\;-f_1(t)\;] & x=l\end{cases}\;,\quad 0\le\,t\,\le T$$

Step 3: Find optimal control $u^{*(k)}$ using conjugate gradient method.

Step 4: Compute $\lambda^*_{Nc/2}=\lambda\,(\,y^{(k)},u^{*(k)}_{Nc/2})\;,\;Z^*_{Nc/2}=Z\,(\,y^{(k)},u^{*(k)}_{Nc/2})$

Step 5: If $\left|\lambda_{Exact}-\lambda^*_{Nc/2}\right|<\;\varepsilon_1\;,\left|Z_{Exact}-Z^*_{Nc/2}\right|<\;\varepsilon_2$ then terminate the procedure, otherwise set $Nc\;=\;Nc\;+\;2\;\;,\;k\;=\;k\;+1$ and go to step 1.

Numerical Results

The problem (1)-(7) is considered as one of the identification problems on definition of unknown coefficients of parabolic quasilinear equation type. The numerical results were carried out for the following example of exact solution, input data:

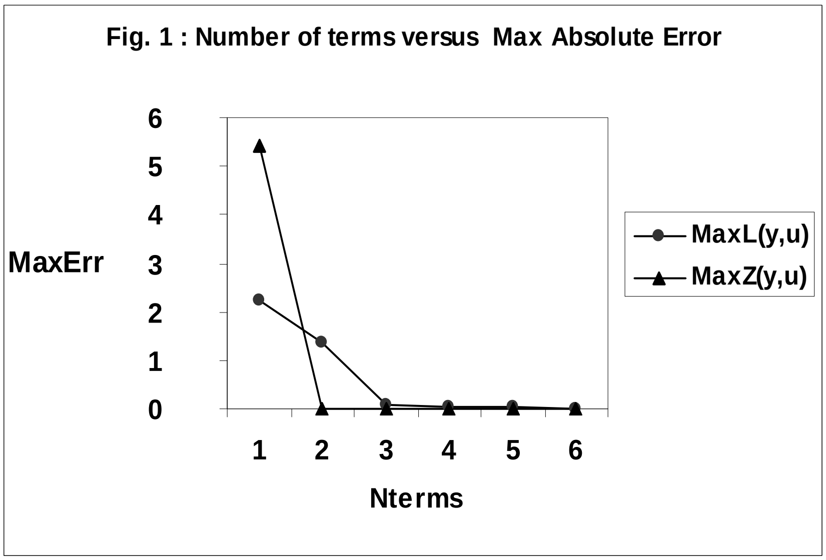
$$y=x+t\,,\,\lambda(y,u)=\tan^{-1}\,(y),Z(y,u)=y^2\,(1-y^2)\,,\,0<x<0.9\,,\,0<t<0.01$$

$$f_0=g_0=\tan^{-1}\,x\,,\;f_1=g_1=\tan^{-1}\,(0.9+t)\,,\;\phi(x)=x,\,F(x,t)=\frac{y}{1+y^2}+\frac{y^3}{1-y^2}$$

In Fig. 1, the curves denoted by MaxL(y,u) and MaxZ(y,u) are the maximum absolute errors

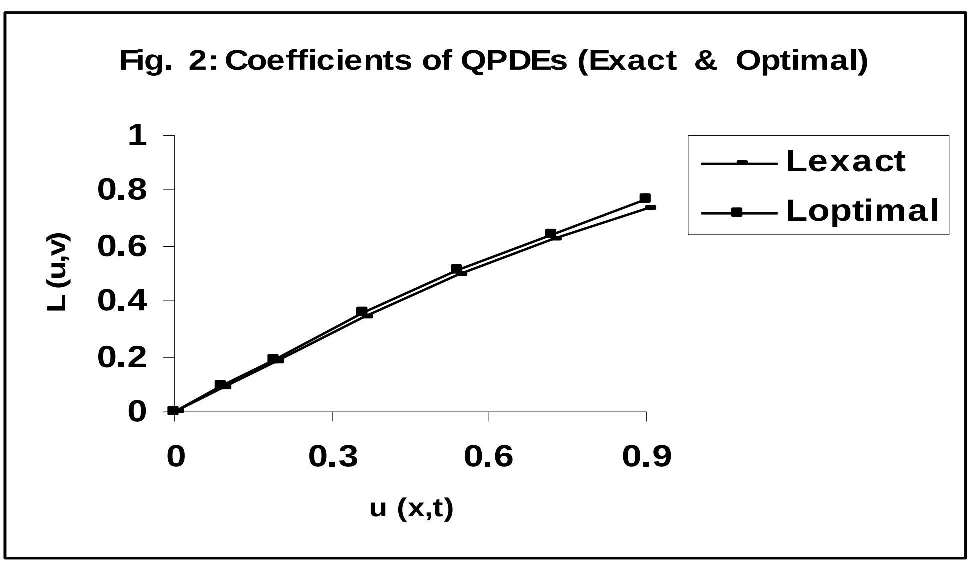
$$MaxL\ (y,u) = \max\ \left|\lambda_{Exact} - \lambda^*_{Nc/2}\right|$$
$$MaxZ\ (y,u) = \max\ \left|Z_{Exact} - Z^*_{Nc/2}\right|$$

It is clear the maximum absolute errors decrease as $Nc \equiv Nterms$ increase.



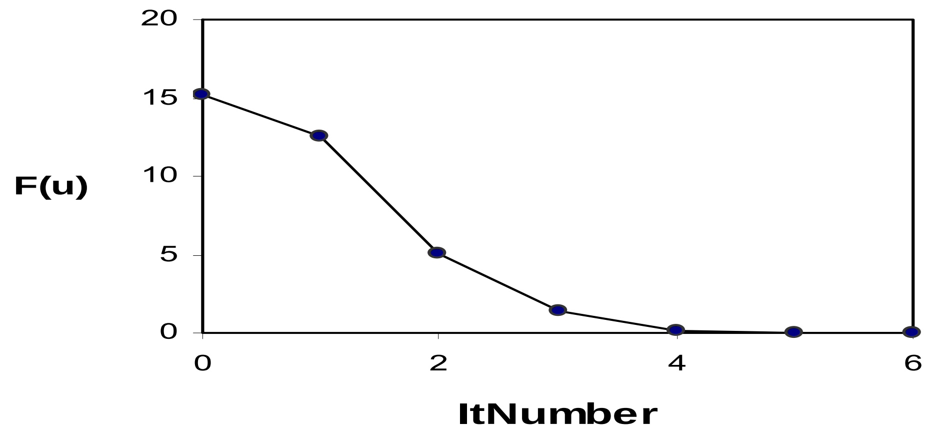
In Fig. 2, the curves denoted by Lexact and Loptimal $(\lambda_{Exact}\ ,\ \lambda_{optimal})$ are the exact values and approximate values with the optimal control $\overset{*}{u}$.

By increasing Nc, $\lambda(y,u)$ will agree with the exact value.



The values of $Y_{\alpha,m}(u) = F(u)$ versus the iteration numbers are displayed in Fig. 3.

Fig. 3 : Iteration Number versus F(u)



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