FUZZY SETS AS A MEAN FOR UNCERTAINTY HANDLING: MATH, APPLIED MATH, HEURISTICS

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Abstract: *Number of Disciplines and Theories changed their status from status of Natural Science discipline to Mathematics. The Theory of Probability is the classical example of that kind. The main privilege of the new, Math, status is the conception of Math truth, which distinguish Math from other theories. Some disciplines, used in Applications, pretended to be Math, not being such. It's entirely true for Fuzzy Subsets Theory with its pretension to be Math and to be exclusive tools in uncertainty handling. Fundamental pretensions of classical Fuzzy subset theory including pretension to be math is discussed as well as the some gaps of the theory are discussed in the article. Statistical interpretation of membership functions is proposed. It is proved, that such the interpretation take place for practically all supporters with minimal constraint on it. Namely, a supporter must be the space with a measure. The interpretation proposed makes it clear the modifying of the classical fuzzy object to correct the gaps. It becomes possible to say about observations of fuzzy subset within the conception of modification and to extend Likelihood method on the new area. Fuzzy likelihood equation adduced as example of new possibilities within approach proposed. One more interpretation for the Fuzzy subset theory is proposed to discuss: multiset theory one.*

Keywords: *f Uncertainty, Plural model of uncertainty, Fuzzy subsets Theory, statistical interpretation of the membership function, modification of Fuzzy subsets, Fuzzy likelihood equation, Multiset theory.*

ACM Classification Keywords: *G.2.m. Discrete mathematics: miscellaneous,G.2.1 Combinatorics. G.3 Probability and statistics, G.1.6. Numerical analysis I.5.1.Pattern Recognition: Models Fuzzy sets; H.1.m. Models and Principles: miscellaneous:*

Introduction

Initially the conception of the Fuzziness planned to be the object of the proposed article. But it became clear after a while that the point of issue ought to be wider. Such expansion must first of all include the discussion about the role of the Fuzziness within the conception of uncertainty. What is the "uncertainty" by itself? Is it mathematics? If not, where ought one to look for the origin of the conception? How does the uncertainty sort with the Fuzziness?

What is primary between these two? There are some more pertinent questions related to place of Math and Applied Math in definitions and applications of the uncertainty and Fuzziness as well as to the role of the Heuristics in Applied researches. Uncertainty surely is the first in discussing about priority within the mentioned pair. For example, Pospelov and his school [Поспелов, 2001] share this opinion. They consider the Fuzziness to be the mean for the uncertainty handling but not vice versa. As to Math, Applied Math it is worth while mentioning in this connection that the Fuzzy subset theory (FzTh) coming into the world due to Lotfi Zadeh [Zadeh, 1965] (see also [Kaufmann, 1982]) was proclaimed to be the mathematical panacea for uncertainty modelling.

Mathematics

There are some principal consideration determined the relations between Math and Empiric experience.

As to the Mathematics by itself. For example Wikipedia [Wikipedia, Math] states the next: "Mathematics is the study of quantity, structure, space, and change. Mathematicians seek out patterns, formulate new conjectures, and establish truth by rigorous deduction from appropriately chosen axioms and definitions." Thus, the specific objects (Math structures) and conception of Math truth (rigorous deduction) are the essence of the Mathematics.

As to the Math structures (see? For example [Донченко, 2009]). When saying "math structure" we ought to understand it to be a set plus "bonds", "relations" between elements of the set. Correspondent "bonds" or "relations" in Math specified by: 1) Math relations (for example "$\leq$" in R^1); 2) functions; 3) operations (for example "+", "$\cdot$" in R^1); 4) collections of the subsets (for example, collection of open subsets, or collection closed subsets or collection of neighbours in R^1); 5) combinations of the four previous. All the Math structures initially have been established for the sets of a numbers of different kinds: integers, real, complex. Then they have been extended on abstract sets. So we have now, for example a structure of metric space (an abstract set plus real valued nonnegative function of two argument with certain properties), structure of group, including affine one (an abstract set plus binary operation with certain properties); structure of linear space (an abstract set with the structure of affine group plus product operations for each of real numbers); structure of Euclidean and Hilbert space (structure of linear space plus non- negative real valued function of two arguments: scalar product); topological space (an abstract set plus an appropriate collection of it subsets, which made possible to define the limit), measurable space (an abstract set plus collection of it subspace, named by σ-algebra), linear topological space and so on. More detailed structure may be considered within the base structure: linear subspace or hyper plane within linear or Euclidean space, subgroup within the group and so on.

Math truth

The fundament of the Math truth is the conception of deducibility. It means that the status of truth (proved statement) has the statement which is terminal in the specially constructed sequence of statements, which called its proof. The peculiarity in sequence constructing means, that a next one in it produced by previous by special admissible rules (deduction rules) from initial admissible statements (axioms and premises of a theorem). As a rule, corresponded admissible statements have the form of equations with the formulas in both its sides. So, each next statement in the sequence-proof of the terminal statement is produced by previous member of sequence (equation) by changing some part of formulas in left or right it side on another: from another side of equations-axioms or equations premises. The specification of the restrictions on admissible statements and the deduction rules are the object of math logic.

Applied Mathematics

The main aim of Applied Math (AppMa) is the description the real object under consideration by Math. This means that the object as "a structure" is represented by the means of Math structure. I.e. the main parts of the object under modelling and principal bonds, ties, relation between them are represented be the means of Math structurization. It is necessary condition for the math modelling to have apt interpretation for a correspondent Math objects and objects under observation. Such interpretations for example, for the function and its derivatives are correspondingly the path and speed for. Integration and differentiation are the means to represent the relation between speed and path. Likewise, a frequencies that or those groups of the result in a sequence of observations are interpreted as a probabilities and vice versa. Surely that or those interpretations can be applied under certain restrictions. So we can't investigate discrete systems by the means of differential equations or apply probabilistic method out of fulfilling stability frequencies low. The main aim of the Math description, the Math modelling of the real object, is to take the advantages establishing true statements for the apt Math object(target statement), which represents real object: for its Math model, with the following interpretation of the correspondent statements. So, if the model of the real object is equation, the target math statement is the statement about its decision and following interpretation of the decision for the real object. So, the next tree-step procedure is the essence of the AppMa. 1. Math decrypting the object on the base of the available knowledge about the real object under consideration and with the help of an apt interpretation. 2. Establishing of math true for the apt (target) statements within the Math model. 3) Interpretation of the target math statement for the real object. The first and the last steps are impossible without interpretation. The availability of interpretation is principal for applied math. Thus, interpretation plus math rigorous truth is the essence of the applied math. So, for example, numerology is not Math and AppMa, because it does not appeal to target (Math true) statements.

Heuristics

There some more means for the investigation, which uses Math on that or this stage or investigation but do not satisfy three stage procedure of AppMa. It may be designated by "intellectual calculus" as some people do. But it is reasonable on my mind to use an apt old good word "heuristic". Indeed, by [Wikipedia: heuristics] "heuristic or heuristics (from the Greek "Εὑρίσκω" for "find" or "discover") refers to experience-based techniques for problem solving, learning, and discovery. Heuristic methods are used to speed up the process of finding a good enough solution, where an exhaustive search is impractical. Examples of this method include using a "rule of thumb", an educated guess, an intuitive judgment, or common sense". We mention opportunely the authors of the "heuristic" from Polya [Polya,1945] through A.Newell&J.C. Shaw& H.A.Simon[Newell& Shaw& Simon,1962] to D. Kahneman [Wikipedia: Kahneman].

Uncertainty

It is common place for the investigators to say or to use the expression "modelling under uncertainties". It is also generally recognized that theory of probability is the classical mean for uncertainty handling when such uncertainty is shown as randomness. Determination of randomness appeals to the notion of experiment (observation, trail, test, sometime – stochastic experiment). So, understanding what the randomness is makes it necessary to look into the conception of "experiment".

Experiment

As the analysis of the numerous sources on Theory of Probability and Math Statistics [Донченко 2009], notion of experiment in them is associated with something, named conditions (condition of experiment), under which phenomena is investigated, and something, that appears under the conditions: named the results of experiment.

So, as in [Донченко 2009] "experiment" is proposed to be considered the pair *(c, y)*: *c*- conditions of experiment (observation, trail, test), y – result of experiment. Henceforth Y_c for the fixed condition *c* will denote the set of all possible that may appear in the experiments under conditions $c \in C$. Generally speaking Y_c is not singleton.

It is reasonable to mark out in a condition c variational, controlled, part x: $x \in R^p$ as a rule, and part f, which is invariable by default in a sequence of experiment. Condition c under such approach is denoted be the pair: c=(x, f), $x \in X \subseteq R^p$.

Sequence of Experiment and their registration

If there are n experiments, then their registration is the sequence

$$(c_i, y_i), c_i \in C, y_i \in Y_{c_i}, i = \overline{1,n}. \quad (1)$$

Different variants of (1) can be implemented in practice

$$((x_i, f), y_i), x_i \in X \subseteq R^p, y_i \in Y_{x_i}, i = \overline{1,n}, \quad (2)$$

$$(x_i, y_i), x_i \in X \subseteq R^p, y_i \in Y_{x_i}, i = \overline{1,n}, \quad (3)$$

$$y_i, y_i \in Y_{c_i}, i = \overline{1,n}, \quad (4)$$

$$y_i, y_i \in Y, i = \overline{1,n} \quad (5)$$

It is obvious, that (5) is equivalent to (1) when all conditions are the same:

$$c_i \equiv c, i = \overline{1,n}.$$

But if it is not so , then

$$Y = \bigcup_{i=1}^{n} Y_{c_i} \neq \text{Singleton}$$

Randomness as a classical example of Uncertainty

Randomness in introduced above designation means firstly, that the results of experiment do not determined by conditions $c \in C$ definitely, i.e.

$$Y_c \neq \text{Singleton}, c \in C. \quad (5)$$

And, secondly, the observations satisfy stability frequency low. This mean: 1) in a sequence of experiment with fixed conditions c frequency of each collection of possible results from Y_c turn to some limit value; 2) the limit value does not depend from the sequence of observations and characterize the phenomenon under consideration. In the Theory of Probability Y_c is called Space of elementary events and is denoted by Ω. Correspond experiment is often called stochastic experiment.

Plural model of Uncertainty

As the randomness is special kind of uncertainty, so the definition of uncertainty one ought to look for in the conception of experiment. Then the natural definition of uncertainty is coincides with the first part of randomness and described by equity (5). Thus, uncertainty is defined on the base of the experiment and classifies certain relation between conditions of experiments *c* and correspond results *y* of it. This relation is stated in (5). We will name such conception of uncertainty by Plural Model of Uncertainty (PluMoU).

Mathematical means for uncertainty handling

There are comparatively few math tools for uncertainty handling. Having no possibility to discuss the theme in detail, we note, that these, namely, are: 1)Theory of probability; 2)inverse problem; 3) maxmin method; 4) Hough Transform; 5) Multisets Theory; 6) Fuzzy(?) Theory; 7) combination of 1)-6) issues. The last point needs some additional explanation in order to embed FzTh in PluMoU. Such embedding become feasible on the base of two possible interpretation of FzTh: within Theory of Probability and Multisets theory.

Fuzzy Theory and statistical interpretation of membership function

Fuzzy set $\underline{\mathrm{A}}$, subset to be more precise (Kaufmann, 1982), as the object in mathematics is nothing more, but the graphic of real valued function μ on an abstract crisp (usual) set *E* (henceforth - supporter of the Fuzzy subset). There is additional constraint on the value of this function, named membership function in Fuzzy theory: its values are bounded by the segment $[0,1]$:

$$\mu : E \to [0,1]\,,\ \underline{\mathrm{A}} = \{(\mathrm{e}, \mu_{\underline{\mathrm{A}}}(\mathrm{e})) :\ \mathrm{e} \in \mathrm{E})\}\,.$$

There are no objections. The definition is perfect but trivial. There great many functions in mathematics, there great many graphics and there are no pretensions of the Fuzzy theory.

Some lacks of the Fuzzy Theory

As it was mentioned about there are several Math tools for uncertainty handling. All of them are well-grounded Math. So, FzTh is not exclusive in pretension on uncertainty handling Also the attention was drawn earlier to the importance of the interpretation for the Applied Math unlike from fundamental. As to FzTh the lack objective interpretation is rather painful problem. The absence of its own set theory as well as a Fuzzy logic is the problem waiting for its solutions. There some steps relating logic (see, for example, [Hajek, 1998F], [Hajek, 1998]). But the problem of interpretation in this case must be solved also. The importance of apt interpretation may be brightly demonstrated on history of the modal logics.

There nothing like axiomatic set theory in FzTh even in naive, Kantor's sense. Particularly, such axiom of paramount importance, known as abstraction [Stoll, 1960] or separation [Kuratovski, Mostowski,1967] principle, is out of consideration. Implementation a variant of this axiom in FzTh would help to close the "object" problem. Indeed, as is well known, the axiom under consideration establishes the correspondence between classical (crisp) subsets and the properties of the elements of the universal set – namely, predicates on the universal crisp set. So, classical predicate have its object of characterization: the correspond set, determined by abstraction axiom. In FzTh changing binary predicates by membership functions forgot to define another element in the pair (predicate, set). Consequence the object of fuzzy characterization was lost. By the way, Multiset theory (some words below) with its technique could help in solving this problem.

It is interesting, that in obvious examples of membership functions out of the FzTh such objects are the intrinsic to the definition of the correspondent objects. Namely, such examples are the generalized variants of logit - and probit (GeLoPr) – regressions, transition matrix for the Markov's chains and Bayesian nets are the mentioned examples.

Natural examples of membership function: Generalized variants of logit - and probit regression

As to these examples, then GeLoPr describes the dependence of the frequencies (probabilities) of the certain event A from the real valued vector under certain parameterization:

$$P\{A \mid H_x\} = G(\beta^T \begin{pmatrix} 1 \\ x \end{pmatrix}),$$

$$\beta \in R^{n-1}, \beta^T = (\beta_{0,}, \dots, \beta_{n-1,}), x \in R^n,$$

where G – distribution function F(z), $x \in R^1$ or correspond tail: 1-F(z) for the scalar distribution.

In this example GeLoPr $\mu(x)=P\{A \mid H_x\}, x \in R^{n-1} = E$ as a function of $x \in R^{n-1}$ is a membership function in the classical FzTh, which corresponds to the certain object, intrinsic for the theory: event A. We would remind, that the event A, mentioned above, describe the presence of certain property in an observation $(x, y), y \in \{0,1\}$. The value 1 for y means the fulfilling and 0 - not fulfilling the property in the observation.

Natural examples of membership function: Markov chain

A transition matrix for the Markov's chain $(\xi_n, n \in \mathbb{N})$, with states set $\wp = \{S_1, ..., S_M(...)\}$ is the $M \times M$ matrix $P = (p_{ij})$ of conditional probabilities:

$$p_{ij} = P\{\xi_{n+1} = S_j \mid \xi_n = S_i\}, i, j = \overline{1, M}$$

Each column with number $j = \overline{1, M}$ of the matrix defines membership function $\mu_j, j = \overline{1, M}$ on $E = \wp$:

$$\mu_j(S_i) = p_{ij} = P\{\xi_{n+1} = S_j \mid \xi_n = S_i\}, j = \overline{1, M}, \quad (6)$$

$$S_i \in \wp = E$$

In each of the M membership functions $\mu_j(S), S \in \wp = E, j = \overline{1, M}$ there are intrinsic objects of fuzzy characterization. Namely these are, correspondingly, $\{\xi_{n+1} = S_j\}, j = \overline{1, M}$.

It is interesting, that it is naturally to consider a (6) to be a “full system” of membership functions: a collection of functions $\mu_j, j = \overline{1, M}$ on E for which

for any $e \in E$

$$\sum_{j=1}^{M} \mu_j(e) = 1, e \in E$$

Natural examples of membership function: Bayesian nets

Any Bayesian net is in the essence a directed weighted graph associated with the probabilistic objects. But if in classic probabilistic graph the weights prescribed to the edges with one and the same head-nodes, in Bayesian – to the one with the same tail-nodes. Thus, the collection of the probabilities is associated with each node: the probabilities, which weight the nodes predecessors. So, correspond probabilities (conditional by its nature) define a membership function.

Probabilistic Interpretation membership function

This subsection deals with the probabilistic interpretation for the classical variant of the FzTh (Donchenko, 1998, 3). Two variant of a supporter E are considered below: discrete and non-discrete. Discrete case is the one which fully illustrates the situation. Namely, each membership function of a fuzzy subset is represented by a system of conditional probabilities of a certain events relatively complete collections of the sets $H_e, e \in E$. Saying "complete collection" we consider the collection $H_e, e \in E$ be the partition of the space of elementary events Ω for a basic probability space

Probabilistic Interpretation membership function: discrete supporter

The main result of the subsection is represented by theorem 1[Donchenko, 1998, 3].

Theorem 1. For any classical Fuzzy Set $(E, \mu_{\underline{A}}(e))$ with discrete support E there exists such discrete probability space

$$(\Omega, B_\Omega, P),$$

event

$$A \in B_\Omega$$

and complete collection of events

$$H_e : H_e \in B_\Omega, e \in E$$

within this probability space such that membership function $\mu_{\underline{A}}(e)$ is represented by the system of conditional probabilities in the next form:

$$\mu_{\underline{A}}(e) = P(A \mid H_e), e \in E. \qquad (7)$$

Theorem 2. For any complete collection of Fuzzy subsets $(E, \mu_{A_i}(e)), i = \overline{1,n}$, with the one and the same supporter E there exists:

discrete probability space (Ω, B_Ω, P);

collection of the evens $A_i \in B_\Omega, i = \overline{1,n}$;

complete collection of the events $H_e : H_e \in B_\Omega, e \in E$,within the probability space (Ω, B_Ω, P), such, that all of the membership functions $\mu_{A_i}(e), e \in E, i = \overline{1,n}$, are simultaneously represented as the systems of conditional probabilities in the next way:

$$\mu_{\underline{A_i}}(e) = P(A_i \mid H_e), e \in E, i = \overline{1,n} .$$

Probabilistic Interpretation membership function: non discrete supporter

The result of the previous subsection may be extended noticeably to non-discrete case if the supporter E possesses certain structure, namely, if it is the space with a measure [Donchenko, 1998, 3].

Theorem 3. Given the:

$(E, \Im, m)$ - is the space with a measure;

$(E, \mu_{\underline{A_i}}(e)), i = \overline{1,n}$, is the complete collection on Fuzzy subsets with the same supporter E;

all of the membership functions $\mu^{(A_i)}(e), i = \overline{1,n}$, are $\Im, £,$ - measurable ($£$ – Borel σ-algebra on R^1),

then, there exist:

probability space (Ω, B_Ω, P),

ξ- discrete random S_p – valued random variable on (Ω, B_Ω, P), where S_p is any n-element set with the elements, say, $S_i, i = \overline{1,n}$;

η random E – valued random variable on (Ω, B_Ω, P) such, that for any $i = \overline{1,n}$

$$\mu^{(A_i)}(e) = P\{\xi = S_i \mid \eta = e\} ,$$

where

$$P\{\xi = S_i \mid \eta\}$$

– conditional distribution of random variable(r.v) ξ respectively r.v. η.

The conditional distribution is regular: for any $e \in E$ $P\{B \mid \eta = e\}$ is a probability respectively B.

Remark on proof. The proof is the result of extending the ideas of the previous theorems but embodied by the application another technique: technique of conditional distribution. The proof being technically complicated is omitted.

Remark 1. There are obvious objects of uncertainty characterization within the theorems 1-3.

Modified Definition of Fuzzy Sets

The way for the solving the problem of constructing the analogue of the separation principle may be on the author opinion the straight reference on the object or property described uncertainly. This reference ought to be reflected evidently in the definition of the membership function:

$$\mu^{(T)}(e), e \in E ,$$

where T – correspondent property (predicate) on certain set U. The last is the set of "uncertain characterization". It may coincides with the E. So $\mu^{\{T\}}(e)$ would be "uncertain characterization"of the property T or corresponding crisp subset $P_T \subseteq U$. The last transition is possible due the separation principle for the crisp sets. Two membership functions $\mu^{\{T_1\}}(e)$ and $\mu^{\{T_2\}}(e)$ with $T_1 \neq T_2$, would specify two different Fuzzy sets, even if they are equal as the function of $e, e \in E$.

Definition. The pair

$$(E, \mu^{(T)}(e))$$

or

$$(E, \mu^{(P_T)}(e))$$

is called the modified Fuzzy subset (MoF) with E as a supporter, which uncertainly describe crisp T on U (or correspondent crisp subset, $P_T \subseteq U$, where U- the "universal" crisp set of "uncertain characterization"), if :

E – is the abstract crisp set, which is referenced to as a supporter;

T - is a crisp predicate onU , correspondingly, P_T - crisp subset ofU, which corresponds to T;

$\mu^{(T)}(e) \in [0,1]$ – function of two arguments: $e, e \in E$ and T from the set of all crisp predicates on universal crisp U.

The function

$$\mu^{(T)}(e), e \in E$$

just as in classical theory of Fuzzy sets will be referenced to as membership function, with adding that it characterizes uncertainly property T (or correspondent subset P_T).

Remark 2. Obviously, statistical interpretation of the theorems 1-3 is applicable to MoF.

Observations of the Modified Fuzzy Sets

The modification of the definition of Fuzzy set introduced earlier in the paper imparts the objectivity to the Fuzzy sets and it is possible now to say about observations of Fuzzy sets for modified ones (Donchenko, 2004). It’s very important ontological aspect for mathematical modeling using Fuzzy sets. The observation of modified Fuzzy sets is the pair $(e, T(e)) - e, e \in E$ – element from the supporter and T(e) is the predicate value on this element. Namely, e is the element, displayed in observation and $T(e)$ is the fixed information about fulfilling the property T in the observation, specified by $e \in E$. It is just in such a way the observations are interpreting in the logit- and probit – regressions and in its generalizations.

So the observation sample is $(e_i, t_i), t_i = T(e_i), i = \overline{1, n}$. One can say about independent observation within statistical interpretation.

Likelihood method for the Modified Fuzzy Sets

Statistical interpretation a membership function grant to say about extension of statistical MLM for estimating fuzzy parameter just as it takes place in the regressions mentioned above.

Indeed, let

$$\mu^{(T)}(e), e \in E$$

-MoF with membership function from parametric collection of membership functions

$$\mu^{(T)}(e) = \mu(e, \beta), \beta \in R^p .$$

Let $(e_i, t_i), i = \overline{1, n}$ independent observation of MoF. We determine "Fuzzy Likelihood function" $FL(\beta)$) by the relation

$$FL(\beta) = \prod_{i=1}^{n} \mu^{t_i}(e_i, \beta)(1 - \mu(e_i, \beta))^{1-t_i} .$$

Correspondingly, we denote by

$$fl(\beta) = \ln FL(\beta) = \sum_{i=1}^{n} t_i \ln \mu(e_i, \beta) +$$

$$+\sum_{i=1}^{n} (1 - t_i) \ln(1 - \mu(e_i, \beta))$$

- logarithmic "Fuzzy Likelihood function".

Just as it is in statistic likelihood estimation

$$\hat{\mu}^{(T)}(e) = \mu(e, \hat{\beta}) ,$$

where

$$\hat{\beta} = \arg \max_{\beta \in R^p} FL(\beta) .$$

Just as in Statistics if $\mu^{(T)}(e) = \mu(e, \beta)$ necessary conditions is the

$$\frac{\partial FL(\beta)}{\partial \beta} = 0$$

or

$$\frac{\partial \mathrm{fl}(\beta)}{\partial \beta} = 0 .$$

The last equation is equivalent the fist one under additional restriction, that the set of zeroes of $\mu(\mathrm{e},\beta), \beta \in \mathrm{R}^{\mathrm{p}}$ respectively $\beta \in \mathrm{R}^{\mathrm{p}}$ is the same for all $\mathrm{e} \in \mathrm{E}$.

The equations of necessary conditions it is naturally to reference to as "fuzzy likelihood equations".

Theorem 4. Under all necessary restrictions "fuzzy likelihood equations" are of the next form

$$\sum_{\mathrm{i}=1}^{\mathrm{n}} \frac{\mathrm{t_i} - \mu(\mathrm{e_i},\beta)}{\mu(\mathrm{e_i},\beta)((1-\mu(\mathrm{e_i},\beta)))} \frac{\partial \mu(\mathrm{e_i},\beta)}{\partial \beta_{\mathrm{j}}} = 0,$$

$$\mathrm{j} = \overline{1,\mathrm{p}}, \beta = \begin{pmatrix} \beta_1 \\ \cdots \\ \beta_{\mathrm{p}} \end{pmatrix} \in \mathrm{R}^{\mathrm{p}} .$$

Experts estimating can be used too by combining LSM and MLM.

Multisets Theory

Multisets (see,for example, reviews: [Blizard, 1989], [Буй, Богатирьова 2010]), is the Math answer for necessity to describe sets which elements with may "repeat". Thus originally conception of multiset implement the idea of repetition rep(u)for elements u from subset D of certain universal set U. Which are the sets D and U , and, correspondingly, rep(u) , depends on peculiarities of applied problem. So, for example, D can be a set of the answers for this or that call in the Internet, відповідей and rep(u) – number of repetition for each record. There is natural way to implement the idea of repetition: to provide each $\mathrm{u} \in \mathrm{D}$ with number or repetition $\mathrm{n_u} : \mathrm{n_u} \in \{1,2,\ldots,\mathrm{n},\ldots\} \equiv \mathrm{N}^+$.

So, we got the first variant for multiset determining 1.We will call by multiset the set of the pairs $\bigcap_{u \in D} \{(u, n_u)\}, n_u \in N^+, u \in D \subseteq U$ for any subset D of certain universal set U . We will call D to be the base of multiset and n_u -multiplicity or repetition factor. This terms will use in all variants of multiset definitins below in evident way. We will denote multiset with base D by $D^{(ms)}$.

Thus, multiset $D^{(ms)}$ is the usual set D with "comments" n_u to its elements.

2. Within the frame of the second definition multiset for any subset D of certain universal set U is the transformation $\alpha : D \to N^+$, defined for any $u \in D$ (see, for example, [Петровский, 2002], [Редько, 2001]) . Equivalence of the first and second determination is evident: $\alpha(u) = n_u, u \in U$. One ought to remark that in second variant the relation function substitutes the set.

3. Third variant: $D^{(ms)}$ for $D \subseteq U$ is the pair $D^{(ms)} \equiv (D, \alpha), \forall D \subseteq U, \forall \alpha : D \to N^+$, α is defined on all elements of D. Thus, in this variant multiset is the pair: set D –"comment" α .

When it necessary we will refer on the components of the multiset-pair $D_{ms} = (D, \alpha)$ n evident way correspondingly by D_α ,and α_D as well as by $D_{D^{(ms)}}, \alpha_{D^{(ms)}} : D = D_{D^{(ms)}}, \alpha_{D^{(ms)}}(u) = \alpha(u), u \in U$.

Natural set terminology take place for the multisets: for the standard operations ("$\bigcup$","$\bigcap$") and for standard relation: "$\subseteq$" . We will denote them for multiset correspondingly "$\bigcup_{ms}$ ","$\bigcap_{ms}$ "" $\subseteq_{ms}$ " .

We will define them

$$\forall D_1^{(ms)} = (D_1, \alpha_1), D_2^{(ms)} = (D_2, \alpha_2) : D_i \subseteq U, i = 1,2$$

by the relations, correspondingly:

1. $$D_1^{(ms)} \subseteq D_2^{(ms)} \Leftrightarrow (D_1 \subseteq D_2 \ \& \ \alpha_1 \le \alpha_2)$$

2. $$D_1^{(ms)} \bigcup_{ms} D_2^{(ms)} \equiv (D_1 \bigcup D_2, \max(\alpha_1, \alpha_2))$$

3. $$D_1^{(ms)}{}_{ms} \bigcap D_2^{(ms)} \equiv (D_1 \bigcap D_2, \min(\alpha_1, \alpha_2))$$

As to operation "$\overline{}$", then it is necessary to "cut" N^+ to $N_M^+ = \{1, 2, \ldots, M\}$ leaving all the rest of the determinations unchangeable. Then $\overline{D^{(ms)}} = \overline{(D, \alpha)}$ is determined by the relation

$$\overline{D^{(ms)}} = (D, M - \alpha)$$

Characteristic function $\chi_{D^{(ms)}}(u)$ (see, for example, [Buy, Bogatyreva, 2010]) is convenient in multiset handling. It is determined by the relation

$$\chi_{D^{(ms)}}(u) = \begin{cases} \alpha(u), u \in D \\ 0, \quad u \notin D \end{cases}$$

Namely, characteristic function is extension of repetition factor or multiplicity on the universal set U.

The role of characteristic functions in multiset theory is fixed by the equivalency in the determination of set operations and order described by the next relations

1. $(D_1^{(ms)} \subseteq D_2^{(ms)}) \Leftrightarrow (\chi_{D_1^{(ms)}} \le \chi_{D_2^{(ms)}})$

2. $\chi_{D_1^{(ms)} \cup D_2^{(ms)}} = \max(\chi_{D_1^{(ms)}}, \chi_{D_2^{(ms)}})$,

3. $\chi_{D_1^{(ms)} \cap D_2^{(ms)}} = \min(\chi_{D_1^{(ms)}}, \chi_{D_2^{(ms)}})$.

Multisets Theory and Fuzziness

It is evidently that in multiset theory repetition factor is the "absolute" variant of membership function. Saying so, we mean absolute and relative frequency. Even more, in the variant of using N_M^+ we get pure membership function by dividing repetition factor α by M. But there are essential differences between these two theories: all membership functions in FzTh are referenced to one and the same E (U in the designations of the multiset theory) and are referenced to particular $D \subseteq U$ in multiset theory. Simple substitution: subsets D instead one and the same universal set in Fzth solve the problem of the object characterization: D is the object. All the rest lacks of the Fzth are also immediately solved the problems: 1)of own set theory with correspond set operations and order; 2) own logic: commonly used mathematical logic; 3) interpretation ($\alpha(u)$ as rep(u)); 4) abstraction axiom: for each $D \subseteq U$ there many possible correspondent α : any of them.

Conclusion

General approach to describing uncertainty was expounded in the paper within conception plurality in understanding uncertainty. The uncertainty is the quality of interaction between researcher and phenomenon within an observation (experiment, trial, and test). Obviously, some formalization for the "observation" is proposed and discussed in the text. The conception of uncertainty proposed make it possible to give for all math means used for uncertainty handling. It is entirely true for then Fuzzy approach after proving principal theorems about statistical interpretation of membership function. Some lacks of the Fuzzy Theory were discussed and some examples and directions of its overcoming were demonstrated. Namely, these were modification, proposed for the membership function and Multiset Theory.

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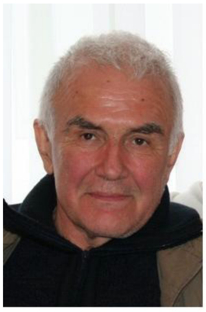

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