REGULAR INVERSIVE POLYTOPES

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***Abstract**: Corresponding to each regular asymptotic polytope* P *in hyperbolic n-space H^n is an isomorphic figure* °P *in inversive ($n-1$)-space I^{n-1} having many related metric properties. A regular inversive polytope* °P *has a* midangle ω *and a* radius ρ, *with 2ω being its* dihedral angle *and 2ρ its* antihedral distance*. The values of ω and ρ are determined for each regular inversive n-polytope.*

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Introduction

The classical real metric spaces (or "spaces of constant curvature") are the *spherical*, *Euclidean*, and *hyperbolic* n-spaces S^n, E^n, and H^n ($n \geq 1$); *elliptic n-space* eP^n results from identifying antipodal points of S^n and has many of the same metric properties. The points at infinity of hyperbolic n-space H^n lie on the *absolute hypersphere*, which has the geometry of a Möbius ($n-1$)-sphere or *inversive* ($n-1$)-space I^{n-1}. Each k-plane of H^n meets the absolute hypersphere in a real inversive ($k-1$)-sphere I^{k-1}, with I^1 being an inversive circle and I^0 a pair of points. Two hyperplanes of H^n may be *intersecting*, meeting in an ($n-2$)-plane; *parallel*, having a single absolute point in common; or *diverging*, with a common perpendicular. The corresponding inversive ($n-2$)-spheres are respectively *separating*, *tangent*, or *separated*.

An *n-polytope* $\mathcal{P}$ is a partially ordered set of j-dimensional "entities" ($-1 \leq j \leq n$), its *j-faces*, satisfying certain incidence conditions, such as those given by McMullen & Schulte (2002, pp. 22–25). A totally ordered subset of j-faces, one of each rank from -1 to n, is a *flag*. When $\mathcal{P}$ is realized as a geometric figure P in some n-dimensional real space, the unique (-1)-face, or *nullity*, can be taken to be the empty set $\varnothing$. The 0-faces are points, the *vertices* of P, and the 1-faces joining adjacent vertices are *edges*. The unique n-face is the *body* of P, essentially its "interior." The ($n-1$)-faces are called *facets*, and the ($n-2$)-faces in which adjacent facets meet are *ridges*. A 2-polytope is a *polygon*, and a 3-polytope is a *polyhedron*. An n-polytope P is *regular* if its symmetry group is transitive on the flags.

Dihedral Angles

It is convenient to denote a regular p-gon of (interior) angle $2\pi/q$ by the extended Schläfli symbol $\{\,p\,\}:q$. Then the polygon is spherical, Euclidean, or hyperbolic according as q is less than, equal to, or greater than $2p/(p-2)$. A regular hyperbolic polygon can be *ordinary*, with all its vertices lying on an ordinary circle, or *asymptotic*, with adjacent sides parallel, so that the vertices all lie on the absolute circle of H^2 and the angles are all zero; a regular asymptotic p-gon thus has the symbol $\{\,p\,\}:\infty$. The center of an ordinary or asymptotic p-gon is an ordinary point. An infinite-sided *apeirogon* $\{\,\infty\,\}:q$ of angle $2\pi/q$ ($q > 2$), whose center is an absolute point, can be inscribed in a horocycle, and an *asymptotic apeirogon* $\{\,\infty\,\}:\infty$ in the absolute circle.

A regular polyhedron whose faces are p-gons, arranged q at a vertex, having dihedral angle $2\pi/r$, is denoted by $\{p, q\}:r$. The polyhedron is spherical, Euclidean, or hyperbolic according as $\sin\pi/p \sin\pi/r$ is greater than, equal to, or less than $\cos\pi/q$, and it is asymptotic when the vertex section $\{q\}:r$ is Euclidean, i.e., when $r = 2q/(q-2)$. The regular asymptotic polyhedra of H^3 consist of the five convex polyhedra

$$\{3,\ 3\}:6,\quad \{3,\ 4\}:4,\quad \{4,\ 3\}:6,\quad \{3,\ 5\}:{}^{10}\!/\!_3,\quad \{5,\ 3\}:6,$$

the four star polyhedra

$$\{{}^5\!/\!_2,\ 5\}:{}^{10}\!/\!_3,\quad \{5,\ {}^5\!/\!_2\}:10,\quad \{{}^5\!/\!_2,\ 3\}:6,\quad \{3,\ {}^5\!/\!_2\}:10,$$

and the three apeirohedra

$$\{4,\ 4\}:4,\quad \{3,\ 6\}:3,\quad \{6,\ 3\}:6.$$

In similar fashion, a regular 4-polytope with facets $\{p, q\}$, r surrounding each edge, and dihedral angle $2\pi/s$ is denoted by $\{p, q, r\}:s$, the polytope being spherical, Euclidean, or hyperbolic depending on the value of s. It is asymptotic when the vertex section $\{q, r\}:s$ is Euclidean, i.e., when $\sin\pi/q \sin\pi/s = \cos\pi/r$. Analogous criteria can be developed for higher-dimensional regular polytopes

$$\{p,\ q,\ldots,\ u,\ v\}:w.$$

Going in the other direction, a one-dimensional polytope comprises a line segment (or a circular arc) and its two endpoints; the whole figure may be called a *ditel*. This can be a circular ditel $\{\ \}:a$ ($a > 2$) in S^1, a straight ditel $\{\ \}:\infty$ in E^1, an ordinary hyperbolic ditel $\{\ \}:b\mathrm{i}$ ($b > 0$) in H^1, or an asymptotic ditel $\{\ \}:0$ (an entire hyperbolic line with its two absolute points). Each finite ditel $\{\ \}:a$, $\{\ \}:\infty$, or $\{\ \}:b\mathrm{i}$ has a unique midpoint, halfway between the endpoints. An asymptotic ditel, however, does not have a well-defined midpoint.

Each vertex of a regular asymptotic n-polytope P in hyperbolic n-space H^n lies on the absolute hypersphere, and each j-face ($1 \le j \le n-1$) lies in a unique j-plane. The j-plane of a j-face meets the absolute hypersphere in an inversive $(j-1)$-sphere. The vertices of P and these $(j-1)$-spheres can be taken as the j-faces ($0 \le j \le n-1$) of an isomorphic *regular inversive n-polytope* °P. The (-1)-face of °P is (as usual) the empty set, and the n-face of °P is the whole absolute hypersphere, regarded as an inversive $(n-1)$-sphere I^{n-1}.

The *dihedral angle* between two adjacent facets of the inversive n-polytope °P is, in general, the angle between two $(n-2)$-spheres on I^{n-1}, which is the same as the dihedral angle between the corresponding facets of the asymptotic n-polytope P. For $n = 1$, P is an asymptotic ditel $\{\ \}:0$, and °P is an inversive *dyad* $°\{\ \}:0$, whose two facets are points, the "angle" between which is infinite. For $n = 2$, P is a regular asymptotic p-gon $\{p\}:\infty$, and °P is a *regular inversive p-gon* $°\{p\}:\infty$, adjacent facets of which are tangent point-pairs (i.e., they have one point in common) on an inversive circle, the angle between which is zero. For $n \ge 3$, the dihedral angle between adjacent facets of a regular inversive n-polytope °P is the positive angle between two separating $(n-2)$-spheres on an inversive $(n-1)$-sphere.

For $n \ge 2$, joining the center of a regular asymptotic n-polytope P to any ridge, or $(n-2)$-face, determines a *median hyperplane* of P. The angle between the median hyperplane and either of the facets that meet at the ridge is the *midangle*, half the dihedral angle. For both P and the corresponding regular inversive polytope °P, it often turns out to be simpler to work with the midangle ω rather than the dihedral angle 2ω.

Metric Formulas

In the Beltrami–Klein model for hyperbolic n-space H^n, the ordinary points of H^n are represented by the interior of an oval $(n-1)$-quadric Φ in projective n-space P^n, with the points of Φ itself representing the absolute hypersphere. Let the points X and hyperplanes $\check{U}$ of P^n have homogeneous coordinates

$$(\!(x)\!) = (x_0,\ x_1, \ldots,\ x_n) \quad \text{and} \quad [\![u]\!] = [u_0,\ u_1, \ldots,\ u_n]$$

with $(\!(x)\!)$ treated as a row and $[\![u]\!]$ as a column. Point X lies on hyperplane $\check{U}$, written $X \diamond \check{U}$, whenever $(\!(x)\!)[\![u]\!] = 0$. Then the $(n-1)$-quadric Φ can be taken to be the locus of self-conjugate points, or the envelope of self-conjugate hyperplanes, in an absolute *hyperbolic polarity* defined by dual bilinear forms

$$(\!(x\ \ y)\!) = (\!(x)\!)H(\!(y)\!)^{\vee} \quad \text{and} \quad [\![u\ \ v]\!] = [\![u]\!]^{\vee}H^{-1}[\![v]\!].$$

Here H is a symmetric $(n+1) \times (n+1)$ matrix congruent to the pseudo-identity matrix $I_{n,1} = \backslash 1, \ldots, 1, -1\backslash$, and the caron denotes the transpose, making $(\!(y)\!)^{\vee}$ a column and $[\![u]\!]^{\vee}$ a row. We may, for instance, take H to be the diagonal matrix $\backslash -1,\ 1, \ldots,\ 1\backslash$, so that the absolute hypersphere $(\!(x\ \ x)\!) = 0$ has the equation

$$x_1^2 + \cdots + x_n^2 = x_0^2.$$

Every ordinary point X with coordinates $(\!(x)\!)$ has $(\!(x\ \ x)\!) < 0$, and every ordinary hyperplane $\check{U}$ with coordinates $[\![u]\!]$ has $[\![u\ \ u]\!] > 0$. The *discriminant* of two ordinary hyperplanes $\check{U}$ and $\check{V}$ may be defined by

$$\|u\ \ v\| = [\![u\ \ u]\!][\![v\ \ v]\!] - [\![u\ \ v]\!]^2,$$

and the hyperplanes are then

intersecting if $\|u\ \ v\| > 0$,

parallel if $\|u\ \ v\| = 0$,

diverging if $\|u\ \ v\| < 0$.

When H^n is taken to have constant curvature -1, we have simple expressions for distances and angles (cf. Coxeter 1998, pp. 209–210). In place of the Euclidean distance $|XY|$, the hyperbolic distance between two ordinary points X and Y is given by

$$]XY[\ = \cosh^{-1} \frac{|(\!(x\ \ y)\!)|}{\sqrt{-(\!(x\ \ x)\!)}\sqrt{-(\!(y\ \ y)\!)}}. \tag{1}$$

The angle between two intersecting or parallel hyperplanes $\check{U}$ and $\check{V}$ is given by

$$(\check{U}\check{V}) = \cos^{-1} \frac{|[\![u\ \ v]\!]|}{\sqrt{[\![u\ \ u]\!]}\sqrt{[\![v\ \ v]\!]}}, \tag{2}$$

and the minimum distance between two diverging hyperplanes $\check{U}$ and $\check{V}$ by

$$)\check{U}\check{V}(\ = \cosh^{-1} \frac{|[\![u\ \ v]\!]|}{\sqrt{[\![u\ \ u]\!]}\sqrt{[\![v\ \ v]\!]}}. \tag{3}$$

The distance between a point X and a hyperplane $\check{U}$ is given by

$$]X\check{U}(\;=\; \sinh^{-1} \frac{|(\!(x)\!)[\![u]\!]|}{\sqrt{-(\!(x\ x)\!)}\sqrt{[\![u\ u]\!]}}. \tag{4}$$

Given any hyperplane $\check{U}$ and a point X not on $\check{U}$, a line through X and one of the absolute points of $\check{U}$ is parallel to $\check{U}$ in that direction. Following Lobachevsky, the angle between the perpendicular from X to $\check{U}$ and the parallel is called the *angle of parallelism* for the distance $x =]X\check{U}($ and is given by

$$\Pi(x) \;=\; \cos^{-1} \tanh x \;=\; 2 \tan^{-1} e^{-x}. \tag{5}$$

As x increases from zero to infinity, $\Pi(x)$ decreases from $\pi/2$ to 0.

A projective hyperplane $\check{U}$, with coordinates $[\![u]\!]$, meets the $(n-1)$-quadric Φ, i.e., the absolute hypersphere of H^n regarded as the inversive $(n-1)$-sphere I^{n-1}, in an inversive $(n-2)$-sphere $\mathring{u}$, a *hypersphere* of I^{n-1}, which is

real if $[\![u\ u]\!] > 0$,

degenerate if $[\![u\ u]\!] = 0$,

imaginary if $[\![u\ u]\!] < 0$.

Taking the discriminant of two real or degenerate hyperspheres $\mathring{u}$ and $\mathring{v}$ to be the discriminant $\|u\ v\|$ of the corresponding hyperplanes $\check{U}$ and $\check{V}$, we find that the hyperspheres are

separating if $\|u\ v\| > 0$,

tangent if $\|u\ v\| = 0$,

separated if $\|u\ v\| < 0$.

Applied to point-pairs $\ddot{u} = \{U_1, U_2\}$ and $\ddot{v} = \{V_1, V_2\}$ on an inversive circle, separation has the usual meaning associated with cyclic order (Coxeter 1966a, p. 218; 1998, pp. 22–23). That is, $\ddot{u}$ and $\ddot{v}$ are separating if $U_1U_2 \,//\, V_1V_2$, tangent if they have a point in common, and separated otherwise. For $n \geq 3$, two hyperspheres of I^{n-1} are separating, tangent, or separated according as they intersect in a real, degenerate, or imaginary $(n-3)$-sphere.

The angle between two separating hyperspheres $\mathring{u}$ and $\mathring{v}$ of I^{n-1}, which is the same as the angle between the corresponding intersecting hyperplanes of H^n, is given by

$$(\mathring{u}\mathring{v}) \;=\; \cos^{-1} \frac{|[\![u\ v]\!]|}{\sqrt{[\![u\ u]\!]}\sqrt{[\![v\ v]\!]}}. \tag{6}$$

Two separated hyperspheres $\mathring{u}$ and $\mathring{v}$ of I^{n-1} have an analogous *inversive distance* (Coxeter 1966b; 1998, pp. 292–298); this is the same as the minimum distance between the corresponding diverging hyperplanes of H^n and is given by

$$)\mathring{u}\mathring{v}(\;=\; \cosh^{-1} \frac{|[\![u\ \ v]\!]|}{\sqrt{[\![u\ \ u]\!]}\sqrt{[\![v\ \ v]\!]}}. \tag{7}$$

The last two formulas are especially relevant to the properties of regular inversive polytopes.

Antihedral Distances

If P is a regular n-polytope in a real metric space, the distance from the center O of the body of P to one of its vertices is the *circumradius* ${}_0R$, and the perpendicular distance from O to (the center of) a facet is the *inradius* ${}_{n-1}R$. When P is centrally symmetric, the *antihedral distance* between a pair of opposite facets is twice the inradius of P. Although an odd polygon $\{\,p\,\}:q$ or a regular simplex $\{\,3^{n-1}\,\}:w$ is not centrally symmetric, the regular compound $\{\{\,p\,\}\}:q$ or $\{\{\,3^{n-1}\,\}\}:w$ of two such polygons or simplexes in dual positions is. The distance between two opposite facets of the compound, one belonging to each component, can be taken as the antihedral distance of the polytope. The inradius of a regular asymptotic apeirotope is infinite, but there is no antihedral distance, since there are no opposite facets.

The *radius* ρ of a regular inversive polytope °P and its *antihedral distance* 2ρ are respectively the inradius and the antihedral distance of the corresponding regular asymptotic polytope P. When ρ is finite, the antihedral distance of °P is the inversive distance between two separated hyperspheres of I^{n-1}, opposite facets either of °P or of a regular compound of two °P's. For an inversive apeirotope, ρ is infinite. The *subradius* σ of °P is the radius of a facet of °P.

The inradius ${}_0R$ of an asymptotic ditel $\{\,\}:0$ or the radius ρ of an inversive dyad $°\{\,\}:0$ is infinite. For a regular inversive p-gon $°\{\,p\,\}:\infty$, ρ is the distance for which the angle of parallelism is π/p, and it follows from (5) that $\tanh\,\rho = \cos\,\pi/p$. Values for particular polygons are given in the table below, where we write τ and $\bar{\tau}$ for the golden-section number $\tfrac{1}{2}(\sqrt{5}+1)$ and its inverse $\tfrac{1}{2}(\sqrt{5}-1)$.

Table 1. *Regular Inversive Polygons*

Polygon	ω	$2\tanh\rho$
$°\{\,3\,\}:\infty$	0	1
$°\{\,4\,\}:\infty$	0	$\sqrt{2}$
$°\{\,5\,\}:\infty$	0	τ
$°\{\,5/2\,\}:\infty$	0	$\bar{\tau}$
$°\{\,6\,\}:\infty$	0	$\sqrt{3}$
$°\{\,p\,\}:\infty$	0	$2\cos\,\pi/p$
$°\{\,\infty\,\}:\infty$	0	2

With the help of some hyperbolic trigonometry, we obtain a simple formula for the radius ρ of a regular inversive polyhedron $°\{p, q\}:r$ in terms of its subradius σ and midangle ω:

$$\tanh \rho = \sinh \sigma \tan \omega = \frac{\tan \pi/r}{\tan \pi/p}. \tag{8}$$

The particular cases are listed in the following table.

Table 2. *Regular Inversive Polyhedra*

Polyhedron	$\tan \omega$	$\tanh \rho$
$°\{3, 3\}:6$	$\frac{1}{3}\sqrt{3}$	$\frac{1}{3}$
$°\{3, 4\}:4$	1	$\frac{1}{3}\sqrt{3}$
$°\{4, 3\}:6$	$\frac{1}{3}\sqrt{3}$	$\frac{1}{3}\sqrt{3}$
$°\{3, 5\}:\frac{10}{3}$	$\sqrt{(3+4\tau)/5}$	$\sqrt{(3+4\tau)/15}$
$°\{5, 3\}:6$	$\frac{1}{3}\sqrt{3}$	$\sqrt{(3+4\tau)/15}$
$°\{\frac{5}{2}, 5\}:\frac{10}{3}$	$\sqrt{(3+4\tau)/5}$	$\frac{1}{5}\sqrt{5}$
$°\{5, \frac{5}{2}\}:10$	$\sqrt{(3-4\bar{\tau})/5}$	$\frac{1}{5}\sqrt{5}$
$°\{\frac{5}{2}, 3\}:6$	$\frac{1}{3}\sqrt{3}$	$\sqrt{(3-4\bar{\tau})/15}$
$°\{3, \frac{5}{2}\}:10$	$\sqrt{(3-4\bar{\tau})/5}$	$\sqrt{(3-4\bar{\tau})/15}$
$°\{4, 4\}:4$	1	1
$°\{3, 6\}:3$	$\sqrt{3}$	1
$°\{6, 3\}:6$	$\frac{1}{3}\sqrt{3}$	1

From the relationship $\tanh \rho = \sinh \sigma \tan \omega$, which holds for all regular inversive n-polytopes with $n \geq 3$, we obtain the parameters for the seventeen regular inversive 4-polytopes $°\{p, q, r\}:s$, as given in the following table. The last figure is an apeirotope. Irrational values of s are rounded to four decimal places.

Table 3. *Regular Inversive* 4-*Polytopes*

4-Polytope	tan ω	tanh ρ
$°\{3,\ 3,\ 3\} : 5.1043$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{4}$
$°\{3,\ 3,\ 4\} : 3.2885$	$\sqrt{2}$	$\frac{1}{2}$
$°\{4,\ 3,\ 3\} : 5.1043$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}$
$°\{3,\ 4,\ 3\} : 4$	1	$\frac{1}{2}\sqrt{2}$
$°\{3,\ 3,\ 5\} : 2.6051$	τ^2	$\frac{1}{4}\sqrt{2}\tau^2$
$°\{5,\ 3,\ 3\} : 5.1043$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{4}\sqrt{2}\tau^2$
$°\{\frac{5}{2},\ 5,\ 3\} : 3.0884$	τ	$\frac{1}{2}\tau$
$°\{3,\ 5,\ \frac{5}{2}\} : 5.6751$	$\bar{\tau}$	$\frac{1}{2}\tau$
$°\{5,\ \frac{5}{2},\ 5\} : 3.0884$	τ	$\frac{1}{2}\tau$
$°\{\frac{5}{2},\ 3,\ 5\} : 2.6051$	τ^2	$\frac{1}{2}$
$°\{5,\ 3,\ \frac{5}{2}\} : 8.6103$	$\bar{\tau}^2$	$\frac{1}{2}$
$°\{\frac{5}{2},\ 5,\ \frac{5}{2}\} : 5.6751$	$\bar{\tau}$	$\frac{1}{2}\bar{\tau}$
$°\{3,\ \frac{5}{2},\ 5\} : 3.0884$	τ	$\frac{1}{2}\bar{\tau}$
$°\{5,\ \frac{5}{2},\ 3\} : 5.6751$	$\bar{\tau}$	$\frac{1}{2}\bar{\tau}$
$°\{\frac{5}{2},\ 3,\ 3\} : 5.1043$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{4}\sqrt{2}\bar{\tau}^2$
$°\{3,\ 3,\ \frac{5}{2}\} : 8.6103$	$\bar{\tau}^2$	$\frac{1}{4}\sqrt{2}\bar{\tau}^2$
$°\{4,\ 3,\ 4\} : 3.2885$	$\sqrt{2}$	1

There are six regular inversive 5-polytopes $°\{p,\ q,\ r,\ s\} : t$, whose midangles ω and radii ρ can be determined from the following table. The last three are apeirotopes.

Table 4. *Regular Inversive* 5-*Polytopes*

5-Polytope	tan ω	tanh ρ
$°\{3,\ 3,\ 3,\ 3\} : 4.7668$	$\frac{1}{5}\sqrt{15}$	$\frac{1}{5}$
$°\{3,\ 3,\ 3,\ 4\} : 3$	$\sqrt{3}$	$\frac{1}{5}\sqrt{5}$
$°\{4,\ 3,\ 3,\ 3\} : 4.7668$	$\frac{1}{5}\sqrt{15}$	$\frac{1}{5}\sqrt{5}$
$°\{4,\ 3,\ 3,\ 4\} : 3$	$\sqrt{3}$	1
$°\{3,\ 3,\ 4,\ 3\} : 3$	$\sqrt{3}$	1
$°\{3,\ 4,\ 3,\ 3\} : 4$	1	1

For $n \geq 6$, there are just four regular asymptotic polytopes in hyperbolic n-space H^n: the asymptotic versions of the regular *n-simplex* and the dual *n-orthoplex* (cross polytope) and *n-orthotope* (block polytope)

$$\{3^{n-1}\}:a_n, \quad \{3^{n-2},\ 4\}:b_n, \quad \{4,\ 3^{n-2}\}:a_n,$$

and the asymptotic *orthic n-apeirotope* (grid apeirotope)

$$\{4,\ 3^{n-3},\ 4\}:b_n.$$

Each of these figures has a corresponding regular inversive polytope in I^{n-1}. The parameters a_n and b_n are defined by

$$\tan\frac{\pi}{a_n} = \sqrt{\frac{n-2}{n}} \quad \text{and} \quad \tan\frac{\pi}{b_n} = \sqrt{n-2}. \tag{9}$$

Note that $\lim_{n\to\infty} a_n = 4$ and $\lim_{n\to\infty} b_n = 2$. The following table gives the midangle ω and radius ρ of each regular inversive n-polytope ($n \geq 6$).

Table 5. *Regular Inversive n-Polytopes*

n-Polytope	$\tan\omega$	$\tanh\rho$
$°\{3,\ 3,\ \ldots,\ 3,\ 3\}:a_n$	$\sqrt{\frac{n-2}{n}}$	$\frac{1}{n}$
$°\{3,\ 3,\ \ldots,\ 3,\ 4\}:b_n$	$\sqrt{n-2}$	$\frac{1}{\sqrt{n}}$
$°\{4,\ 3,\ \ldots,\ 3,\ 3\}:a_n$	$\sqrt{\frac{n-2}{n}}$	$\frac{1}{\sqrt{n}}$
$°\{4,\ 3,\ \ldots,\ 3,\ 4\}:b_n$	$\sqrt{n-2}$	1

Regular inversive polytopes with identical vertex sections have the same midangle ω, and dual polytopes

$$°\{p,q,\ldots,u,v\}:w \quad \text{and} \quad °\{v,u,\ldots,q,p\}:o$$

have the same radius ρ. Also, except for polygons $°\{p\}:\infty$ with more than six sides, $\tan\,\omega$ and $\tanh\,\rho$ either are rational numbers or have expressions involving nothing worse than nested square roots.

Conformal Models

The Euclidean plane can be given the topology of a sphere by means of a "one-point compactification," i.e., by adjoining a single point at infinity $\dot{O}$ that lies on every line. If such extended lines are treated as infinite (or "great") circles on the same footing as ordinary (or "small") circles, and if we allow a circle-preserving transformation, or *circularity*, to move the point $\dot{O}$, the resulting "inversive plane" or *parabolic sphere* $\dot{\mathrm{S}}^2$ provides a conformal model for the inversive sphere I^2. A circularity is called a *homography* or an *antihomography* according as it preserves or reverses orientation; homographies are also known as Möbius transformations.

An ordinary point with Cartesian coordinates (x, y) has *parabolic coordinates*

$$(\tfrac{1}{2}(x^2+y^2+1),\ x,\ y,\ \tfrac{1}{2}(x^2+y^2-1)),$$

while the exceptional point $\dot{O}$ is

$$(1,\ 0,\ 0,\ 1).$$

A small circle with center (h, k) and radius r has parabolic coordinates

$$[\tfrac{1}{2}(r^2-h^2-k^2-1),\ h,\ k,\ \tfrac{1}{2}(r^2-h^2-k^2+1)],$$

The equation of a small circle is $(x-h)^2+(y-k)^2 = r^2$. A great circle, i.e., a line, with inclination θ and displacement c has coordinates

$$[c,\ \sin\theta,\ -\cos\theta,\ -c],$$

with $0 \le \theta < \pi$. The equation of a line is $y\cos\theta = x\sin\theta + c$; if $\theta \ne \pi/2$, the line has slope $m = \tan\theta$ and y-intercept $b = c\sec\theta$. Parabolic coordinates for points and circles of $\dot{\mathrm{S}}^2$ (and nonzero scalar multiples thereof) can be taken as homogeneous coordinates for points and circles of I^2.

An inversive circle $\mathring{c}$, with coordinates $[\![c]\!] = [c_0,\ c_1,\ c_2,\ c_3]$, is real, degenerate, or imaginary according as the quadratic form

$$[\![c\ \ c]\!] = c_1^2+c_2^2+c_3^2-c_0^2$$

Is positive, zero, or negative. In the nondegenerate cases an *inversion* in $\mathring{c}$ is the involutory circularity $\mathrm{I}^2 \to \mathrm{I}^2$ induced by the pseudo-orthogonal *inversion matrix*

$$C = \frac{1}{[\![c\ \ c]\!]}\begin{pmatrix} c_0^2+c_1^2+c_2^2+c_3^2 & -2c_0c_1 & -2c_0c_2 & -2c_0c_3 \\ 2c_1c_0 & -c_0^2-c_1^2+c_2^2+c_3^2 & -2c_1c_2 & -2c_1c_3 \\ 2c_2c_0 & -2c_2c_1 & -c_0^2+c_1^2-c_2^2+c_3^2 & -2c_2c_3 \\ 2c_3c_0 & -2c_3c_1 & -2c_3c_2 & -c_0^2+c_1^2+c_2^2-c_3^2 \end{pmatrix} \tag{10}$$

(cf. Schwerdtfeger 1962, pp. 117–118). Each point $(\!(p)\!) = (p_0,\ p_1,\ p_2,\ p_3)$ is interchanged with the point $(\!(p)\!)C$. If $[\![c\ \ c]\!] > 0$, this is a *hyperbolic inversion*, leaving all points on the real circle $\mathring{c}$ invariant and taking each circle orthogonal to $\mathring{c}$ into itself. If $[\![c\ \ c]\!] < 0$, the circularity is an *elliptic inversion* in the imaginary circle $\mathring{c}$, leaving no real points invariant. Every circularity of I^2 is the product of (at most four) hyperbolic inversions.

If we now fix a *central* inversion in a particular real inversive circle Ω, say the unit circle or the x-axis, we obtain a conformal model for the metric *hyperbolic sphere* $\ddot{\mathrm{S}}^2$ (cf. Johnson 1981, pp. 452–454). A point on one side of Ω inverts into a point on the other side, and such *antipodal* point-pairs represent the ordinary points of the *hyperbolic plane* H^2. The self-antipodal points on the "equator" Ω are the absolute points of H^2. Inversive circles orthogonal to Ω ("great" circles of $\ddot{\mathrm{S}}^2$) represent ordinary lines of H^2. A circularity of I^2 that takes Ω into itself is an isometry of H^2. The mapping $\mathrm{I}^2 \to \mathrm{H}^2$ preserves angular measure, and the minimum distance between two diverging lines in H^2 is the same as the inversive distance between the representative separated circles in I^2 (cf. Coxeter 1998, pp. 308–311). Note that a regular inversive polygon $°\{\,p\,\}:\infty$ whose vertices all lie on Ω is the trace on Ω of a regular asymptotic polygon $\{\,p\,\}:\infty$.

The mapping just described can be made one-to-one by identifying antipodal points of $\ddot{\mathrm{S}}^2$ or, equivalently, by restricting the domain to points on one side of Ω, e.g., the interior of the unit circle or points with positive y-coordinates, with hyperbolic lines represented by inversive circular arcs instead of whole circles. In this manner we obtain Poincaré's "conformal disk" and "upper half-plane" models for the hyperbolic plane.

As a model for the inversive sphere I^2, the completed Euclidean plane can be replaced by the *elliptic sphere* S^2, taken as the unit sphere in Euclidean 3-space. A conformal mapping from the "equatorial plane" $z = 0$ to S^2 can be achieved by stereographic projection, in which a line through an arbitrary point $(\,x,\ y,\ 0\,)$ and the "north pole" $(\,0,\ 0,\ 1\,)$ meets the sphere again in the point $(\,\xi,\ \eta,\ \zeta)$, with $(\,0,\ 0,\ 1\,)$ itself corresponding to the exceptional point $\dot{O}$ (cf. Schwerdtfeger 1962, pp. 22–29). The relationship between two-dimensional Cartesian coordinates $(\,x,\ y\,)$ and spherical coordinates $(\,\xi,\ \eta,\ \zeta)$ is given by

$$\xi = \frac{2x}{x^2 + y^2 + 1}, \qquad \eta = \frac{2y}{x^2 + y^2 + 1}, \qquad \zeta = \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1}. \tag{11}$$

The parabolic coordinates $(\,\tfrac{1}{2}(x^2 + y^2 + 1\,),\ x,\ y,\ \tfrac{1}{2}(x^2 + y^2 - 1\,)\,)$ of a point can be replaced by *normalized coordinates* $(\,1,\ \xi,\ \eta,\ \zeta)$, with $\xi^2 + \eta^2 + \zeta^2 = 1$.

A regular asymptotic polyhedron $\{\,p,\ q\,\}:r$ of finite inradius can be represented in the Beltrami–Klein model for H^3 by a regular Euclidean polyhedron $\{\,p,\ q\,\}:r'$ inscribed in the unit sphere. Although it does not preserve angles between ordinary planes, so that $r' \neq r$, the Beltrami–Klein model is conformal on the absolute sphere. Thus the inversive circles in which adjacent face-planes of $\{\,p,\ q\,\}:r'$ meet the sphere intersect in an angle of $2\pi/r$, producing a regular inversive polyhedron $°\{\,p,\,q\,\}:r$.

The above procedures can be extended to construct conformal Euclidean and spherical models for hyperbolic n-space and inversive $(n-1)$-space and for regular inversive n-polytopes $°\{\,p\,,\,q\,,\,.\,.\,.\,,\ u,\ v\}:w$.

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