

CONES OF WEIGHTED QUASIMETRICS, WEIGHTED QUASIHYPERMETRICS AND OF ORIENTED CUTS

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Abstract: We show that the cone of weighted n -point quasi-metrics $WQMet_n$, the cone of weighted quasi-hypermetrics $WHyp_n$ and the cone of oriented cuts $OCut_n$ are projections along an extreme ray of the metric cone Met_{n+1} , of the hypermetric cone Hyp_{n+1} and of the cut cone Cut_{n+1} , respectively. This projection is such that if one knows all faces of an original cone then one knows all faces of the projected cone.

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1 Introduction

Oriented (or directed) distances are encountered very often, for example, these are one-way transport routes, a river with quick flow and so on.

The notions of directed distances, quasi-metrics and oriented cuts are generalizations of the notions of distances, metrics and cuts, which are central objects in graph theory and combinatorial optimization.

Quasi-metrics are used in semantics of computations (see, for example, [Se97]) and in computational geometry (see, for example, [AACMP97]). Oriented distances have been used already by Hausdorff in 1914, see [Ha14].

In [CMM06], an example of directed metric derived from a metric is given. Let d be a metric on a set $V \cup \{0\}$, where 0 is a distinguished point. Then a quasi-metric q on the set V is given as

$$q_{ij} = d_{ij} + d_{i0} - d_{j0}.$$

This quasi-metric belongs to a special important subclass of quasi-metrics, namely, to a class of *weighted quasi-metrics*. We show in this paper that *any* weighted quasi-metric is obtained from a metric by this method.

All semi-metrics on a set of cardinality n form a metric cone Met_n . There are two important sub-cones of Met_n , namely, the cone Hyp_n of hypermetrics, and the cone Cut_n of ℓ_1 -metrics. These three cones form the following nested family $Cut_n \subseteq Hyp_n \subseteq Met_n$, see [DL97].

In this paper we introduce a special space Q_n , called a space of weighted quasi-metrics. We define in this space a cone $WQMet_n$. Elements of this cone satisfy triangle and non-negativity inequalities. Among extreme rays of the cone $WQMet_n$ there are rays spanned by ocut vectors, i.e. incidence vectors of oriented cuts.

We define in the space Q_n a cone $OCut_n$ of oriented cuts as the cone hull of ocut vectors. Elements of the cone $OCut_n$ are weighted quasi- ℓ -metrics.

Let metrics of the cone Met_{n+1} are defined on the set $V \cup \{0\}$. The cut-cone Cut_{n+1} of ℓ_1 -metrics on this set is a cone hull of cut-metrics $\delta(S)$ for all $S \subset V \cup \{0\}$. The cut-metrics $\delta(S)$ are extreme rays of all the three cones Met_{n+1} , Hyp_{n+1} and Cut_{n+1} . In particular, $\delta(\{0\}) = \delta(V)$ is an extreme ray of these three cones.

In this paper, it is shown that the cones $WQMet_n$ and $OCut_n$ are projections of the corresponding cones Met_{n+1} and Cut_{n+1} along the extreme ray $\delta(V)$. We define a cone $WQHyp_n$ of weighted quasi-hypermetrics as projection along $\delta(V)$ of the cone Hyp_{n+1} . So, we obtain a nested family $OCut_n \subseteq WQHyp_n \subseteq WQMet_n$.

Weighted quasi-metrics and other generalizations of metrics are studied, for example, in [DD10] and [DDV11]. The cone and the polytope of oriented cuts are considered in [AM11].

2 Spaces \mathbb{R}^E and \mathbb{R}^{E°

Let V be a set of cardinality $|V| = n$. Let E and E° be sets of all unordered (ij) and ordered ij pairs of elements $i, j \in V$. Consider two Euclidean spaces \mathbb{R}^E and \mathbb{R}^{E° of vectors $d \in \mathbb{R}^E$ and $g \in \mathbb{R}^{E^\circ}$ with coordinates $d_{(ij)}$ and g_{ij} , where $(ij) \in E$ and $ij \in E^\circ$, respectively. Obviously, dimensions of the spaces \mathbb{R}^E and \mathbb{R}^{E° are $|E| = \frac{n(n-1)}{2}$ and $|E^\circ| = n(n-1)$, respectively.

Denote by $(d, t) = \sum_{(ij) \in E} d_{(ij)} t_{(ij)}$ scalar product of vectors $d, t \in \mathbb{R}^E$. Similarly $(f, g) = \sum_{ij \in E^\circ} f_{ij} g_{ij}$ denote scalar product of vectors $f, g \in \mathbb{R}^{E^\circ}$.

Let $\{e_{(ij)} : (ij) \in E\}$ and $\{e_{ij} : ij \in E^\circ\}$ be orthonormal bases of \mathbb{R}^E and \mathbb{R}^{E° , respectively. Then, for $f \in \mathbb{R}^E$ and $q \in \mathbb{R}^{E^\circ}$, we have

$$(e_{(ij)}, f) = f_{(ij)} \text{ and } (e_{ij}, q) = q_{ij}.$$

For $f \in \mathbb{R}^{E^\circ}$, define $f^* \in \mathbb{R}^{E^\circ}$ as follows

$$f_{ij}^* = f_{ji} \text{ for all } ij \in E^\circ.$$

Call a vector g *symmetric* if $g^* = g$, and *antisymmetric* if $g^* = -g$. Each vector $g \in \mathbb{R}^{E^\circ}$ can be decompose into *symmetric* g^s and *antisymmetric* g^a parts as follows:

$$g^s = \frac{1}{2}(g + g^*), \quad g^a = \frac{1}{2}(g - g^*), \quad g = g^s + g^a.$$

Let $\mathbb{R}_s^{E^\circ}$ and $\mathbb{R}_a^{E^\circ}$ be subspaces of symmetric and antisymmetric vectors, respectively. Note that the spaces $\mathbb{R}_s^{E^\circ}$ and $\mathbb{R}_a^{E^\circ}$ are mutually orthogonal. In fact, for $p \in \mathbb{R}_s^{E^\circ}$ and $f \in \mathbb{R}_a^{E^\circ}$, we have

$$(p, f) = \sum_{ij \in E^\circ} p_{ij} f_{ij} = \sum_{(ij) \in E} (p_{ij} f_{ij} + p_{ji} f_{ji}) = \sum_{(ij) \in E} (p_{ij} f_{ij} - p_{ij} f_{ij}) = 0.$$

Hence

$$\mathbb{R}^{E^\circ} = \mathbb{R}_s^{E^\circ} \oplus \mathbb{R}_a^{E^\circ},$$

where \oplus is direct sum.

Obviously, there is an isomorphism φ between the spaces \mathbb{R}^E and $\mathbb{R}_s^{E^\circ}$. Let $d \in \mathbb{R}^E$ have coordinates $d_{(ij)}$. Then we set

$$d^\circ = \varphi(d) \in \mathbb{R}_s^{E^\circ} \text{ such that } d_{ij}^\circ = d_{ji}^\circ = d_{(ij)}.$$

In particular,

$$\varphi(e_{(ij)}) = e_{ij} + e_{ji}.$$

The map φ is invertible. In fact, for $q \in \mathbb{R}_s^{E^\circ}$, we have $\varphi^{-1}(q) = d \in \mathbb{R}^E$ such that $d_{(ij)} = q_{ij} = q_{ji}$. The isomorphism φ will be useful in what follows.

3 Space of weights Q_n^w

One can consider the sets E and E° as sets of edges (ij) and arcs ij of an unordered and ordered complete graphs K_n and K_n° on the vertex set V , respectively. The graph K_n° has two arcs ij and ji between each pair of vertices $i, j \in V$.

It is convenient to consider vectors $g \in \mathbb{R}^{E^\circ}$ as functions on the set of arcs E° of the graph K_n° . So, the decomposition $\mathbb{R}^{E^\circ} = \mathbb{R}_s^{E^\circ} \oplus \mathbb{R}_a^{E^\circ}$ is a decomposition of the space of all functions on arcs in E° onto the spaces of symmetric and antisymmetric functions.

Besides, there is an important direct decomposition of the space $\mathbb{R}_a^{E^\circ}$ of antisymmetric functions onto two subspaces. In theory of electric networks these spaces are called spaces of *tensions* and *flows* (see also [Aig79]).

The tension space relates to *potentials* (or *weights*) w_i given on vertices $i \in V$ of the graph K_n° . The corresponding antisymmetric function g^w is determined as

$$g_{ij}^w = w_i - w_j.$$

It is called *tension* on the arc ij . Obviously, $g_{ji}^w = w_j - w_i = -g_{ij}^w$. Denote by Q_n^w the subspace of \mathbb{R}^{E° generated by all tensions on arcs $ij \in E^\circ$. We call Q_n^w by a *space of weights*.

Each tension function g^w is represented as weighted sum of *elementary potential* functions $p(k)$ for $k \in V$ as follows

$$g^w = \sum_{k \in V} w_k p(k),$$

where

$$p(k) = \sum_{j \in V - \{k\}} (e_{kj} - e_{jk}), \text{ for all } k \in V, \quad (1)$$

are basic functions that generate the space of weights Q_n^w . Hence values of the basic functions $p(k)$ on arcs are as follows

$$p_{ij}(k) = \begin{cases} 1 & \text{if } i = k \\ -1 & \text{if } j = k \\ 0 & \text{otherwise.} \end{cases} \quad (2)$$

We obtain

$$g_{ij}^w = \sum_{k \in V} w_k p_{ij}(k) = w_i - w_j.$$

It is easy to verify that

$$p^2(k) = (p(k), p(k)) = 2(n-1), \quad (p(k), p(l)) = -2 \text{ for all } k, l \in V, k \neq l, \quad \sum_{k \in V} p(k) = 0.$$

Hence there are only $n-1$ independent functions $q(k)$ that generate the space Q_n^w .

Weighted quasimetrics lie in the space $\mathbb{R}_s^{E^\circ} \oplus Q_n^w$ that we denote as Q_n . Direct complements of Q_n^w in $\mathbb{R}_a^{E^\circ}$ and Q_n in \mathbb{R}^{E° is a space Q_n^c of *circuits* (or *flows*).

4 Space of circuits Q_n^c

The space of circuits (or space of flows) is generated by characteristic vectors of oriented circuits in the graph K_n° . Arcs of K_n° are ordered pairs ij of vertices $i, j \in V$. The arc ij is oriented from the vertex i to the vertex j . Recall that K_n° has both the arcs ij and ji for each pair of vertices $i, j \in V$.

Let $G_s \subseteq K_n$ be an undirected subgraph with a set of edges $E(G_s) \subseteq E$. We relate to the undirected graph G_s a directed graph $G \subseteq K_n^\circ$ with the arc set $E^\circ(G) \subseteq E^\circ$ as follows. An arc ij belongs to G , i.e. $ij \in E^\circ(G)$, if and only if $(ij) = (ji) \in E(G)$. This definition implies that the arc ji belongs to G also, i.e. $ji \in E^\circ(G)$.

Let C_s be a circuit in the graph K_n . The circuit C_s is determined by a sequence of distinct vertices $i_k \in V$, where $1 \leq k \leq p$ and p is the length of C_s . Edges of C_s are unordered pairs (i_k, i_{k+1}) , where indices are taken modulo p . By above definition, an *oriented bicircuit* C of the graph K_n° relates to the circuit C_s . Arcs of C are ordered pairs $i_k i_{k+1}$ and $i_{k+1} i_k$, where indices are taken modulo p . Take an orientation of C . Denote by $-C$ the *opposite* circuit with opposite orientation. Denote an arc of C *direct* or *opposite* if its direction coincides with or is opposite to the given orientation of C , respectively. Let C^+ and C^- be subcircuits of C consisting of direct and opposite arcs, respectively.

The following vector f^C is the characteristic vector of the bicircuit C :

$$f_{ij}^C = \begin{cases} 1 & \text{if } ij \in C^+, \\ -1 & \text{if } ij \in C^-, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $f^{-C} = (f^C)^* = -f^C$, and $f^C \in \mathbb{R}_a^{E^\circ}$.

Denote by Q_n^c the space linearly generated by circuit vectors f^C for all bicircuits C of the graph K_n° . It is well known that characteristic vectors of *fundamental* circuits form a basis of Q_n^c . Fundamental circuits are defined as follows.

Let T be a spanning tree of the graph K_n . Since T is spanning, its vertex set $V(T)$ is the set of all vertices of K_n , i.e. $V(T) = V$. Let $E(T) \subset E$ be the set of edges of T . Then any edge $e = (ij) \notin E(T)$ closes a unique path in T between vertices i and j into a circuit C_s^e . This circuit C_s^e is called *fundamental*. Call corresponding oriented bicircuit C^e also *fundamental*.

There are $|E - E(T)| = \frac{n(n-1)}{2} - (n-1)$ fundamental circuits. Hence

$$\dim Q_n^c = \frac{n(n-1)}{2} - (n-1), \text{ and } \dim Q_n + \dim Q_n^c = n(n-1) = \dim \mathbb{R}^{E^\circ}.$$

This implies that Q_n^c is an orthogonal complement of Q_n^w in \mathbb{R}_a° and Q_n in \mathbb{R}^{E° , i.e.

$$\mathbb{R}_a^{E^\circ} = Q_n^w \oplus Q_n^c \text{ and } \mathbb{R}^{E^\circ} = Q_n \oplus Q_n^c = \mathbb{R}^{E_s^\circ} \oplus Q_n^w \oplus Q_n^c.$$

5 Cut and ocut vector set-functions

The space Q_n is generated also by vectors of oriented cuts, which we define in this section.

Each subset $S \subseteq V$ determines cuts of the graphs K_n and K_n° that are subsets of edges and arcs of these graphs.

A $cut(S) \subset E$ is a subset of edges (ij) of K_n such that $(ij) \in cut(S)$ if and only if $|\{i, j\} \cap S| = 1$.

A $cut^\circ(S) \subset E^\circ$ is a subset of arcs ij of K_n° such that $ij \in cut^\circ(S)$ if and only if $|\{i, j\} \cap S| = 1$. So, if $ij \in cut^\circ(S)$, then $ji \in cut^\circ(S)$ also.

An *oriented cut* is a subset $ocut(S) \subset E^\circ$ of arcs ij of K_n° such that $ij \in ocut(S)$ if and only if $i \in S$ and $j \notin S$.

We relate to these three types of cuts characteristic vectors $\delta(S) \in \mathbb{R}^E$, $\delta^\circ(S) \in \mathbb{R}_s^{E^\circ}$, $p(S) \in \mathbb{R}_a^{E^\circ}$ and $c(S) \in \mathbb{R}^{E^\circ}$ as follows.

For $cut(S)$, we set

$$\delta(S) = \sum_{i \in S, j \in \bar{S}} e_{(ij)}, \text{ such that } \delta_{(ij)}(S) = \begin{cases} 1 & \text{if } |\{i, j\} \cap S| = 1 \\ 0 & \text{otherwise,} \end{cases}$$

where $\bar{S} = V - S$. For $cut^\circ(S)$, we set

$$\delta^\circ(S) = \varphi(\delta(S)) = \sum_{i \in S, j \in \bar{S}} (e_{ij} + e_{ji}) \text{ and } p(S) = \sum_{i \in S, j \in \bar{S}} (e_{ij} - e_{ji}).$$

Hence

$$\delta_{ij}^\circ(S) = \begin{cases} 1 & \text{if } |\{i, j\} \cap S| = 1 \\ 0 & \text{otherwise.} \end{cases} \text{ and } p_{ij}(S) = \begin{cases} 1 & \text{if } i \in S, j \notin S \\ -1 & \text{if } j \in S, i \notin S \\ 0 & \text{otherwise.} \end{cases}$$

Note that, for one-element sets $S = \{k\}$, the function $p(\{k\})$ is $p(k)$ of section 2. It is easy to see that

$$(\delta^{\mathcal{O}}(S), p(T)) = 0 \text{ for any } S, T \subseteq V.$$

For the oriented cut $ocut(S)$, we set

$$c(S) = \sum_{i \in S, j \in \bar{S}} e_{ij}.$$

Hence

$$c_{ij}(S) = \begin{cases} 1 & \text{if } i \in S, j \notin S \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, it holds $c(\emptyset) = c(V) = \mathbf{0}$, where $\mathbf{0} \in \mathbb{R}^{E^{\mathcal{O}}}$ is a vector whose all coordinates are equal zero. We have the following equalities

$$c^*(S) = c(\bar{S}), \quad c(S) + c(\bar{S}) = \delta^{\mathcal{O}}(S), \quad c(S) - c(\bar{S}) = p(S) \text{ and } c(S) = \frac{1}{2}(\delta^{\mathcal{O}}(S) + p(S)). \quad (3)$$

Besides, we have

$$c^s(S) = \frac{1}{2}\delta^{\mathcal{O}}(S), \quad c^a(S) = \frac{1}{2}p(S).$$

Recall that a set-function $f(S)$ on all $S \subseteq V$, is called *submodular* if, for any $S, T \subseteq V$, the following *submodular inequality* holds

$$f(S) + f(T) - (f(S \cap T) + f(S \cup T)) \geq 0.$$

It is well known that the vector set-function $\delta \in \mathbb{R}^E$ is submodular (see, for example, [Aig79]). The above isomorphism φ of the spaces \mathbb{R}^E and $\mathbb{R}_s^{E^{\mathcal{O}}}$ implies that the vector set-function $\delta^{\mathcal{O}} = \varphi(\delta) \in \mathbb{R}_s^{E^{\mathcal{O}}}$ is submodular also.

A set-function $f(S)$ is called *modular* if, for any $S, T \subseteq V$, the above submodular inequality holds as equality. This equality is called *modular equality*. It is well known (and can be easily verified) that antisymmetric vector set-function $f^a(S)$ is modular for any oriented graph G . Hence our antisymmetric vector set-function $q(S) \in \mathbb{R}_a^{E^{\mathcal{O}}}$ for the oriented complete graph $K_n^{\mathcal{O}}$ is modular also.

Note that the set of all submodular set functions on a set V forms a cone in the space \mathbb{R}^{2^V} . Therefore the last equality in (3) implies that the vector set-function $c(S) \in \mathbb{R}^{E^{\mathcal{O}}}$ is submodular.

The modularity of the antisymmetric vector set-function $q(S)$ is important for what follows. It is well-known (see, for example, [Bir67]) (and it can be easily verified using modular equality) that any modular set-function $m(S)$ is completely determined by its values on the empty set and on all one-element sets. Hence a modular set-function $m(S)$ has the following form

$$m(S) = m_0 + \sum_{i \in S} m_i,$$

where $m_0 = m(\emptyset)$ and $m_i = m(\{i\}) - m(\emptyset)$. For brevity, we set $f(\{i\}) = f(i)$ for any set function $f(S)$. Since $p(\emptyset) = p(V) = 0$, we have

$$p(S) = \sum_{k \in S} p(k), \quad S \subseteq V, \text{ and } p(V) = \sum_{k \in V} p(k) = 0. \quad (4)$$

Using equations (3) and (4), we obtain

$$c(S) = \frac{1}{2}(\delta^{\mathcal{O}}(S) + \sum_{k \in S} p(k)). \quad (5)$$

Now we show that ocut vectors $c(S)$ for all $S \subseteq V$ linearly generate the space $Q_n \subseteq \mathbb{R}^{E^{\mathcal{O}}}$. The space generated by $c(S)$ consists of the following vectors

$$c = \sum_{S \subseteq V} \alpha_S c(S), \text{ where } \alpha_S \in \mathbb{R}.$$

Recall that $c(S) = \frac{1}{2}(\delta^{\mathcal{O}}(S) + p(S))$. Hence we have

$$c = \frac{1}{2} \sum_{S \subset V} \alpha_S (\delta^{\mathcal{O}}(S) + p(S)) = \frac{1}{2} \sum_{S \subset V} \alpha_S \delta^{\mathcal{O}}(S) + \frac{1}{2} \sum_{S \subset V} \alpha_S p(S) = \frac{1}{2}(d^{\mathcal{O}} + p),$$

where $d^{\mathcal{O}} = \varphi(d)$ for $d = \sum_{S \subseteq V} \alpha_S \delta(S)$. For a vector p we have

$$p = \sum_{S \subset V} \alpha_S p(S) = \sum_{S \subset V} \alpha_S \sum_{k \in S} p(k) = \sum_{k \in V} w_k p(k), \text{ where } w_k = \sum_{V \supset S \ni k} \alpha_S.$$

Since $p_{ij} = \sum_{k \in V} w_k p_{ij}(k) = w_i - w_j$, we have

$$c_{ij} = \frac{1}{2}(d_{ij}^{\mathcal{O}} + w_i - w_j). \quad (6)$$

It is well-known (see, for example, [DL97]) that the vectors $\delta(S) \in \mathbb{R}^E$ for all $S \subseteq V$ linearly generate the full space \mathbb{R}^E . Hence the vectors $\delta^{\mathcal{O}}(S) \in \mathbb{R}_s^{E^{\mathcal{O}}}$ for all $S \subseteq V$ linearly generate the full space $\mathbb{R}_s^{E^{\mathcal{O}}}$.

According to (5), antisymmetric parts of vectors $c(S)$ generate the space Q_n^w . This implies that the space $Q_n = \mathbb{R}_s^{E^{\mathcal{O}}} \oplus Q_n^w$ is generated by $c(S)$ for all $S \subset V$.

6 Properties of the space Q_n

Let $x \in Q_n$ and let f^C be the characteristic vector of a bicircuit C . Since f^C is orthogonal to Q_n , we have $(x, f^C) = \sum_{ij \in C} f_{ij}^C x_{ij} = 0$. This equality implies that each point $x \in Q_n$ satisfies the following equalities

$$\sum_{ij \in C^+} x_{ij} = \sum_{ij \in C^-} x_{ij}$$

for any bicircuits C .

Let $K_{1,n-1} \subset K_n$ be a spanning star of K_n consisting of all $n-1$ edges incident to a vertex of K_n . Let this vertex be 1. Each edge of $K_n - K_{1,n-1}$ has the form (ij) , where $i \neq 1 \neq j$. The edge (ij) closes a fundamental triangle with edges $(1i), (1j), (ij)$. The corresponding bitriangle $T(1ij)$ generates the equality

$$x_{1i} + x_{ij} + x_{j1} = x_{i1} + x_{1j} + x_{ji}.$$

These inequalities were derived by another way in [AM11]. They correspond to fundamental bi-triangles $T(1ij)$, for all $i, j \in V - \{1\}$, and are all $\frac{n(n-1)}{2} - (n-1)$ independent equalities determining the space, where the Q_n lies.

Above coordinates x_{ij} of a vector $x \in Q_n$ are given in the orthonormal basis $\{e_{ij} : ij \in E^{\mathcal{O}}\}$. But, for what follows, it is more convenient to consider vectors $q \in Q_n$ in another basis. Recall that $\mathbb{R}_s^{E^{\mathcal{O}}} = \varphi(\mathbb{R}^E)$. Let, for $(ij) \in E$, $\varphi(e_{(ij)}) = e_{ij} + e_{ji} \in \mathbb{R}_s^{E^{\mathcal{O}}}$ be basic vectors of the subspace $\mathbb{R}_s^{E^{\mathcal{O}}} \subset Q_n$. Let $p(i) \in Q_n^w, i \in V$, be basic vectors (defined in (1)) of the space $Q_n^w \subset Q_n$. Then, for $q \in Q_n$, we set

$$q = q^s + q^a, \text{ where } q^s = \sum_{(ij) \in E} q_{(ij)} \varphi(e_{(ij)}), \quad q^a = \sum_{i \in V} w_i p(i).$$

Now, we obtain an important expression for the scalar product (g, q) of vectors $g, q \in Q_n$. Recall that $(\varphi(e_{(ij)}), p(k)) = ((e_{ij} + e_{ji}), p(k)) = 0$ for all $(ij) \in E$ and all $k \in V$. Hence $(g^s, q^a) = (g^a, q^s) = 0$, and we have

$$(g, q) = (g^s, q^s) + (g^a, q^a).$$

Besides, we have

$$((e_{ij} + e_{ji}), (e_{kl} + e_{lk})) = 0 \text{ if } (ij) \neq (kl), \quad (e_{ij} + e_{ji})^2 = 2,$$

and (see Section 3)

$$(p(i), p(j)) = -2 \text{ if } i \neq j, (p(i))^2 = 2(n-1).$$

Let $v_i, w_i, i \in V$, be weights of the vector g, q , respectively. Then we have

$$(g, q) = 2 \sum_{(ij) \in E} g_{(ij)} q_{(ij)} + 2(n-1) \sum_{i \in V} v_i w_i - 2 \sum_{i \neq j \in V} v_i w_j.$$

For the last sum, we have

$$\sum_{i \neq j \in V} v_i w_j = \left(\sum_{i \in V} v_i \right) \left(\sum_{i \in V} w_i \right) - \sum_{i \in V} v_i w_i.$$

Since weights are defined up to an additive scalar, we can choose weights v_i such that $\sum_{i \in V} v_i = 0$. Then the last sum in the product (g, q) is equal to $-\sum_{i \in V} v_i w_i$. Finally we obtain that the sum of antisymmetric parts is equal to $2n \sum_{i \in V} v_i w_i$. So, for the product of two vectors $g, q \in Q_n$ we have the following expression

$$(g, q) = (g^s, q^s) + (g^a, q^a) = 2 \left(\sum_{(ij) \in E} g_{(ij)} q_{(ij)} + n \sum_{i \in V} v_i w_i \right) \text{ if } \sum_{i \in V} v_i = 0 \text{ or } \sum_{i \in V} w_i = 0.$$

In what follows, we consider inequalities $(g, q) \geq 0$. We can delete the multiple 2, and rewrite such inequality as follows

$$\sum_{(ij) \in E} g_{(ij)} q_{(ij)} + n \sum_{i \in V} v_i w_i \geq 0, \quad (7)$$

where $\sum_{i \in V} v_i = 0$.

Below we consider some cones in the space Q_n . Since the space Q_n is orthogonal to the space of circuits Q_n^c , each facet vector of a cone in Q_n is defined up to a vector of the space Q_n^c . Of course each vector $g' \in \mathbb{R}^{E^c}$ can be decomposed as $g' = g + g^c$, where $g \in Q_n$ and $g^c \in Q_n^c$. Call the vector $g \in Q_n$ *canonical representative* of the vector g' . Usually we will use canonical facet vectors. But sometimes not canonical representatives of a facet vector are useful.

Cones C that will be considered are invariant under the operation $q \rightarrow q^*$, defined in Section 2. In other words, $C^* = C$. This operation changes signs of weights:

$$q_{ij} = q_{(ij)} + w_i - w_j \rightarrow q_{(ij)} + w_j - w_i = q_{(ij)} - w_i + w_j.$$

Let $(g, q) \geq 0$ be an inequality determining a facet F of a cone $C \subset Q_n$. Since $C = C^*$, the cone C has with the facet F also a facet F^* . The facet F^* is determined by the inequality $(g^*, q) \geq 0$.

7 Projections of cones Con_{n+1}

Recall that $Q_n = \mathbb{R}_s^{E^c} \oplus Q_n^w, \mathbb{R}_s^{E^c} = \varphi(\mathbb{R}^E)$ and $\dim Q_n = \frac{n(n+1)}{2} - 1$.

Let $0 \notin V$ be an additional point. Then the set of unordered pairs (ij) for $i, j \in V \cup \{0\}$ is $E \cup E_0$, where $E_0 = \{(0i) : i \in V\}$. Obviously, $\mathbb{R}^{E \cup E_0} = \mathbb{R}^E \oplus \mathbb{R}^{E_0}$ and $\dim \mathbb{R}^{E \cup E_0} = \frac{n(n+1)}{2}$.

The space $\mathbb{R}^{E \cup E_0}$ contains the following three important cones: the cone Met_{n+1} of semi-metrics, the cone Hyp_{n+1} of hyper-semi-metrics and the cone Cut_{n+1} of ℓ_1 -semi-metrics, all on the set $V \cup \{0\}$. Denote by Con_{n+1} any of these cones.

Recall that a semi-metric $d = \{d_{(ij)}\}$ is called *metric* if $d_{(ij)} \neq 0$ for all $(ij) \in E$. For brevity sake, in what follows, we call elements of the cones Con_{n+1} by metrics (or hypermetrics, ℓ_1 -metrics), assuming that they can be semi-metrics.

Note that if $d \in Con_{n+1}$ is a metric on the set $V \cup \{0\}$, then a restriction d^V of d on the set V is a point of the cone $Con_n = Con_{n+1} \cap \mathbb{R}^E$ of metrics on the set V . In other words, we can suppose that $Con_n \subset Con_{n+1}$.

The cones Met_{n+1} , Hyp_{n+1} and Cut_{n+1} contain the cut vectors $\delta(S)$ that span extreme rays for all $S \subset V \cup \{0\}$. Denote by l_0 the extreme ray spanned by the cut vector $\delta(V) = \delta(\{0\})$. Consider a projection $\pi(\mathbb{R}^{E \cup E_0})$ of the space $\mathbb{R}^{E \cup E_0}$ along the ray l_0 onto a subspace of $\mathbb{R}^{E \cup E_0}$ that is orthogonal to $\delta(V)$. This projection is such that $\pi(\mathbb{R}^E) = \mathbb{R}^E$ and $\pi(\mathbb{R}^{E \cup E_0}) = \mathbb{R}^E \oplus \pi(\mathbb{R}^{E_0})$.

Note that $\delta(V) \in \mathbb{R}^{E_0}$, since, by Section 5, $\delta(V) = \sum_{i \in V} e_{(0i)}$. For simplicity sake, define the following vector

$$e_0 = \delta(\{0\}) = \delta(V) = \sum_{i \in V} e_{(0i)}.$$

Recall that the vector e_0 spans the extreme ray l_0 . Obviously, the space \mathbb{R}^E is orthogonal to l_0 , and therefore $\pi(\mathbb{R}^E) = \mathbb{R}^E$.

Let $x \in \mathbb{R}^{E \cup E_0}$. We decompose this point as follows

$$x = x^V + x^0,$$

where $x^V = \sum_{(ij) \in E} x_{(ij)} e_{(ij)} \in \mathbb{R}^E$ and $x^0 = \sum_{i \in V} x_{(0i)} e_{(0i)} \in \mathbb{R}^{E_0}$. The projection π works on basic vectors as follows:

$$\pi(e_{(ij)}) = e_{(ij)} \text{ for } (ij) \in E, \text{ and } \pi(e_{(0i)}) = e_{(0i)} - \frac{1}{n} e_0 \text{ for } i \in V.$$

So, we have

$$\pi(x) = \pi(x^V) + \pi(x^0) = \sum_{(ij) \in E} x_{(ij)} e_{(ij)} + \sum_{i \in V} x_{(0i)} (e_{(0i)} - \frac{1}{n} e_0). \quad (8)$$

It is useful to note that the projection π transforms the positive orthant of the space \mathbb{R}^{E_0} onto the whole space $\pi(\mathbb{R}^{E_0})$.

Now we describe how faces of a cone in the space $\mathbb{R}^{E \cup E_0}$ are projected along one of its extreme rays.

Let l be an extreme ray and F be a face of a cone in $\mathbb{R}^{E \cup E_0}$. Let π be the projection along l . Let $\dim F$ be dimension of the face F . Then the following equality holds

$$\dim \pi(F) = \dim F - \dim(F \cap l). \quad (9)$$

Let $g \in \mathbb{R}^{E \cup E_0}$ be a facet vector of a facet G , and e be a vector spanning the line l . Then $\dim(G \cap l) = 1$ if $(g, e) = 0$, and $\dim(G \cap l) = 0$ if $(g, e) \neq 0$.

Theorem 1. *Let G be a face of the cone $\pi(\text{Con}_{n+1})$. Then $G = \pi(F)$, where F is a face of Con_{n+1} such that there is a facet of Con_{n+1} , containing both F and the extreme ray l_0 spanned by $e_0 = \delta(V)$.*

In particular, G is a facet of $\pi(\text{Con}_{n+1})$ if and only if $G = \pi(F)$, where F is a facet of Con_{n+1} containing the extreme ray l_0 . Similarly, l' is an extreme ray of $\pi(\text{Con}_{n+1})$ if and only if $l' = \pi(l)$, where l is an extreme ray of Con_{n+1} lying in a facet of Con_{n+1} that contains l_0 .

Proof. Let \mathcal{F} be a set of all facets of the cone Con_{n+1} . Then $\cup_{F \in \mathcal{F}} \pi(F)$ is a covering of the projection $\pi(\text{Con}_{n+1})$. By (9), in this covering, if $l_0 \subseteq F \in \mathcal{F}$, then $\pi(F)$ is a facet of $\pi(\text{Con}_{n+1})$. If $l_0 \not\subseteq F$, then there is a one-to-one correspondence between points of F and $\pi(F)$. Hence $\dim \pi(F) = n$, and $\pi(F)$ cannot be a facet of $\pi(\text{Con}_{n+1})$, since $\pi(F)$ fills an n -dimensional part of the cone $\pi(\text{Con}_{n+1})$.

If F' is a face of Con_{n+1} , then $\pi(F')$ is a face of the above covering. If F' belongs only to facets $F \in \mathcal{F}$ such that $l_0 \not\subseteq F$, then $\pi(F')$ lies inside of $\pi(\text{Con}_{n+1})$. In this case, it is not a face of $\pi(\text{Con}_{n+1})$. This implies that $\pi(F')$ is a face of $\pi(\text{Con}_{n+1})$ if and only if $F' \subseteq F$, where F is a facet of Con_{n+1} such that $l_0 \subseteq F$. Suppose that dimension of F' is $n-1$, and $l_0 \not\subseteq F'$. Then $\dim \pi(F') = n-1$. If F' is contained in a facet F of Con_{n+1} such that $l_0 \subseteq F$, then $\pi(F') = \pi(F)$. Hence $\pi(F')$ is a facet of the cone $\pi(\text{Con}_{n+1})$ that coincides with the facet $\pi(F)$.

Now, the assertions of Theorem about facets and extreme rays of $\pi(\text{Con}_{n+1})$ follow. \square

Theorem 1 describes all faces of the cone $\pi(\text{Con}_{n+1})$ if one knows all faces of the cone Con_{n+1} .

Recall that we consider $\text{Con}_n = \text{Con}_{n+1} \cap \mathbb{R}^E$ as a sub-cone of Con_{n+1} , and therefore $\pi(\text{Con}_n) \subset \pi(\text{Con}_{n+1})$. Since $\pi(\mathbb{R}^E) = \mathbb{R}^E$, we have $\pi(\text{Con}_n) = \text{Con}_n$. Let $(f, x) \geq 0$ be a facet-defining inequality of a facet F of the cone Con_{n+1} . Since $\text{Con}_{n+1} \subset \mathbb{R}^E \oplus \mathbb{R}^{E_0}$, we represent vectors $f, x \in \mathbb{R}^{E \cup E_0}$ as $f = f^V + f^0, x = x^V + x^0$, where $f^V, x^V \in \mathbb{R}^E$ and $f^0, x^0 \in \mathbb{R}^{E_0}$. Hence the above facet-defining inequality can be rewritten as

$$(f, x) = (f^V, x^V) + (f^0, x^0) \geq 0.$$

It turns out that the cone Con_{n+1} has always a facet F whose facet vector $f = f^V + f^0$ is such that $f^0 = 0$. Since f^V is orthogonal to \mathbb{R}^{E_0} , the hyperplane $(f^V, x) = (f^V, x^V) = 0$ supporting the facet F contains the whole space \mathbb{R}^{E_0} . The equality $(f^V, x^V) = 0$ defines a facet $F^V = F \cap \mathbb{R}^E$ of the cone Con_n .

Definition. A facet F of the cone Con_{n+1} with a facet vector $f = f^V + f^0$ is called *zero-lifting* of a facet F^V of Con_n if $f^0 = 0$ and $F \cap \mathbb{R}^E = F^V$.

Similarly, a facet $\pi(F)$ of the cone $\pi(\text{Con}_{n+1})$ with a facet vector f is called *zero-lifting* of F^V if $f = f^V$ and $\pi(F) \cap \mathbb{R}^E = F^V$.

It is well-known, see, for example, [DL97], that each facet F^V with facet vector f^V of the cone Con_n can be zero-lifted up to a facet F of Con_{n+1} with the same facet vector f^V .

Proposition 1. *Let a facet F of Con_{n+1} be zero-lifting of a facet F^V of Con_n . Then $\pi(F)$ is a facet of $\pi(\text{Con}_{n+1})$ that is also zero-lifting of F^V .*

Proof. Recall that the hyperplane $\{x \in \mathbb{R}^{E \cup E_0} : (f^V, x) = 0\}$ supporting the facet F contains the whole space \mathbb{R}^{E_0} . Hence the facet F contains the extreme ray l_0 spanned by the vector $e_0 \in \mathbb{R}^{E_0}$. By Theorem 1, $\pi(F)$ is a facet of $\pi(\text{Con}_{n+1})$. The facet vector of $\pi(F)$ can be written as $f = f^V + f'$, where $f^V \in \mathbb{R}^E$ and $f' \in \pi(\mathbb{R}^{E_0})$. Since the hyperplane supporting the facet $\pi(F)$ is given by the equality $(f^V, x) = 0$ for $x \in \pi(\mathbb{R}^{E \cup E_0})$, we have $f' = 0$. Besides, obviously, $\pi(F) \cap \mathbb{R}^E = F^V$. Hence $\pi(F)$ is zero-lifting of F^V . \square

8 Cones $\psi(\text{Con}_{n+1})$

Note that basic vectors of the space $\mathbb{R}^{E \cup E_0}$ are $e_{(ij)}$ for $(ij) \in E$ and $e_{(0i)}$ for $(0i) \in E_0$. Since $\pi(e_0) = \sum_{i \in V} \pi(e_{(0i)}) = 0$, we have $\dim \pi(\mathbb{R}^{E_0}) = n - 1 = \dim Q_n^w$. Recall that $\pi(\mathbb{R}^E) = \mathbb{R}^E$. Hence there is a one-to-one bijection χ between the spaces $\pi(\mathbb{R}^{E \cup E_0})$ and Q_n .

We define this bijection $\chi : \pi(\mathbb{R}^{E \cup E_0}) \rightarrow Q_n$ as follows

$$\chi(\mathbb{R}^E) = \varphi(\mathbb{R}^E) = \mathbb{R}_s^{E^{\mathcal{O}}}, \text{ and } \chi(\pi(\mathbb{R}^{E_0})) = Q_n^w,$$

where

$$\chi(e_{(ij)}) = \varphi(e_{(ij)}) = e_{ij} + e_{ji}, \text{ and } \chi(\pi(e_{(0i)})) = \chi(e_{(0i)} - \frac{1}{n}e_0) = p(i),$$

where $p(i)$ is defined in (1).

Note that $(e_{ij} + e_{ji})^2 = 2 = 2e_{(ij)}^2$ and

$$(p(i), p(j)) = -2 = 2n((e_{(0i)} - \frac{1}{n}e_0), (e_{(0j)} - \frac{1}{n}e_0)), \quad p^2(i) = 2(n-1) = 2n(e_{(0i)} - \frac{1}{n}e_0)^2.$$

Roughly speaking, the map χ is a composition of homotheties that extends vectors $e_{(ij)}$ and $e_{(0i)} - \frac{1}{n}e_0$ up to vectors $e_{ij} + e_{ji}$ and $p(i)$ by the multiples $\sqrt{2}$ and $\sqrt{2n}$, respectively.

Setting $\psi = \chi \circ \pi$, we obtain a map $\psi : \mathbb{R}^{E \cup E_0} \rightarrow Q_n$ such that

$$\psi(e_{(ij)}) = e_{ij} + e_{ji} \text{ for } (ij) \in E, \quad \psi(e_{(0i)}) = p(i) \text{ for } i \in V. \quad (10)$$

Now we show how a point $x = x^V + x^0 \in \mathbb{R}^{E \cup E_0}$ is transformed into a point $q = \psi(x) = \chi(\pi(x)) \in Q_n$. We have $\pi(x) = x^V + \pi(x^0)$, where, according to (8), $x^V = \sum_{(ij) \in E} x_{(ij)} e_{(ij)} \in \pi(\mathbb{R}^E) = \mathbb{R}^E$ and $\pi(x^0) = \sum_{i \in V} x_{(0i)} (e_{(0i)} - \frac{1}{n} e_0) \in \pi(\mathbb{R}^{E_0})$. Obviously, $\chi(x^V + \pi(x^0)) = \chi(x^V) + \chi(\pi(x^0))$, and

$$\psi(x^V) = \chi(x^V) = \sum_{(ij) \in E} x_{(ij)} (e_{ij} + e_{ji}) = \varphi(x^V) = q^s \text{ and } \chi(\pi(x^0)) = \sum_{i \in V} x_{(0i)} p(i) = q^a.$$

Recall that $q^s = \sum_{(ij) \in E} q_{(ij)} (e_{ij} + e_{ji})$ and $q^a = \sum_{i \in V} w_i p(i)$. Hence

$$q_{(ij)} = x_{(ij)}, (ij) \in E, \text{ and } w_i = x_{(0i)}, i \in V. \quad (11)$$

Let $f \in \mathbb{R}^{E \cup E_0}$ be a facet vector of a facet F of the cone Con_{n+1} , $f = f^V + f^0 = \sum_{(ij) \in E} f_{(ij)} e_{(ij)} + \sum_{i \in V} f_{(0i)} e_{(0i)}$.

Let $(f, x) \geq 0$ be the inequality determining the facet F . The inequality $(f, x) \geq 0$ takes on the set $V \cup \{0\}$ the following form

$$(f, x) = \sum_{(ij) \in E} f_{(ij)} x_{(ij)} + \sum_{i \in V} f_{(0i)} x_{(0i)} \geq 0.$$

Since $x_{(ij)} = q_{(ij)}$, $x_{(0i)} = w_i$, we can rewrite this inequality as follows

$$(f, q) = (f^V, q^s) + (f^0, q^a) \equiv \sum_{(ij) \in E} f_{(ij)} q_{(ij)} + \sum_{i \in V} f_{(0i)} w_i \geq 0. \quad (12)$$

Comparing the inequality (12) with (7), we see that a canonical form of the facet vector f is $f = f^s + f^a$, where

$$f_{ij}^s = f_{(ij)}, \text{ for } (ij) \in E, f_{ij}^a = v_i - v_j \text{ where } v_i = \frac{1}{n} f_{(0i)}, i \in V. \quad (13)$$

Theorem 2. *Let F be a facet of the cone Con_{n+1} . Then $\psi(F)$ is a facet of the cone $\psi(Con_{n+1})$ if and only if the facet F contains the extreme ray l_0 spanned by the vector e_0 .*

Let $l \neq l_0$ be an extreme ray of Con_{n+1} . Then $\psi(l)$ is an extreme ray of $\psi(Con_{n+1})$ if and only if the ray l belongs to a facet containing the extreme ray l_0 .

Proof. By Theorem 1, the projection π transforms the facet F of Con_{n+1} into a facet of $\pi(Con_{n+1})$ if and only if $l_0 \subset F$. By the same Theorem, the projection $\pi(l)$ is an extreme ray of $\pi(Con_{n+1})$ if and only if l belongs to a facet containing the extreme ray l_0 .

Recall that the map χ is a bijection between the spaces $\mathbb{R}^{E \cup E_0}$ and Q_n . This implies the assertion of this Theorem for the map $\psi = \chi \circ \pi$. \square

By Theorem 2, the map ψ transforms the facet F in a facet of the cone $\psi(Con_{n+1})$ only if F contains the extreme ray l_0 , i.e. only if the equality $(f, e_0) = 0$ holds. Hence the facet vector f should satisfy the equality $\sum_{i \in V} f_{(0i)} = 0$.

The inequalities (12) give all facet-defining inequalities of the cone $\psi(Con_{n+1})$ from known facet-defining inequalities of the cone Con_{n+1} .

So, we have the following algorithm for to find a list of facets of the cone $\psi(Con_{n+1})$ from a known list \mathcal{L} of facet vectors of the cone Con_{n+1} .

Step 1. Take a facet vector $f = \{f_{(ij)} : (ij) \in E \cup E_0\} \in \mathcal{L}$ of the cone Con_{n+1} , and delete it from \mathcal{L} . Find a point $i \in V \cup \{0\}$ such that $\sum_{k \in V \cup \{0\}} f_{(ik)} = 0$. Go to Step 2.

Step 2. If such a point i does not exist, go to Step 1. Otherwise, make a permutation $i \rightarrow 0, 0 \rightarrow i$, and go to step 3.

Step 3. By formula (13) form a facet vector of the cone $\psi(Con_{n+1})$ from the facet vector f of the cone Con_{n+1} . If \mathcal{L} is not empty, go to Step 1. Otherwise, end.

A proof of Proposition 2 below will be given later for each of the cones Met_{n+1} , Hyp_{n+1} and Cut_{n+1} separately.

Proposition 2. Let F be a facet of Con_{n+1} with facet vector $f = f^V + f^0$ such that $(f^0, e_0) = 0$. Then Con_{n+1} has also a facet F^* with facet vector $f^* = f^V - f^0$.

Proposition 2 implies the following important fact.

Proposition 3. For $q = q^s + q^a \in \psi(Con_{n+1})$, the map $q = q^s + q^a \rightarrow q^* = q^s - q^a$ preserves the cone $\psi(Con_{n+1})$, i.e.

$$(\psi(Con_{n+1}))^* = \psi(Con_{n+1}).$$

Proof. Let F be a facet of Con_{n+1} with facet vector f . By Proposition 2, if $\psi(F)$ is a facet of $\psi(Con_{n+1})$, then F^* is a facet of Con_{n+1} with facet vector f^* . Let $q \in \psi(Con_{n+1})$. Then q satisfies as the inequality $(f, q) = (f^V, q^s) + (f^0, q^a) \geq 0$ (see (12)) so the inequality $(f^*, q) = (f^V, q^s) - (f^0, q^a) \geq 0$. But it is easy to see that $(f, q) = (f^*, q^*)$ and $(f^*, q) = (f, q^*)$. This implies that $q^* \in \psi(Con_{n+1})$. \square

Call a facet G of the cone $\psi(Con_{n+1})$ *symmetric* if $q \in G$ implies $q^* \in G$. Call a facet of $\psi(Con_{n+1})$ *asymmetric* if it is not symmetric.

The assertion of the following Proposition 4 is implied by the equality $(\psi(Con_{n+1}))^* = \psi(Con_{n+1})$.

Proposition 4. Let $g \in Q_n$ be a facet vector of an asymmetric facet G of the cone $\psi(Con_{n+1})$, and let $G^* = \{q^* : q \in G\}$. Then G^* is a facet of $\psi(Con_{n+1})$, and g^* is its facet vector.

Recall that Con_{n+1} has facets, that are zero-lifting of facets of Con_n . Call a facet G of the cone $\psi(Con_{n+1})$ *zero-lifting* of a facet F^V of Con_n if $G = \psi(F)$, where F is a facet of Con_{n+1} which is zero-lifting of F^V .

Proposition 5. Let $g \in Q_n$ be a facet vector of a facet G of the cone $\psi(Con_{n+1})$. Then the following assertions are equivalent:

- (i) $g = g^*$;
- (ii) the facet G is symmetric;
- (iii) $G = \psi(F)$, where F is a facet of Con_{n+1} which is zero-lifting of a facet F^V of Con_n .
- (iv) G is a zero-lifting of a facet F^V of Con_n .

Proof. (i) \Rightarrow (ii). If $g = g^*$, then $g = g^s$. Hence $q \in G$ implies $(g, q) = (g^s, q) = (g^s, q^s) = (g, q^*) = 0$. This means that $q^* \in G$, i.e. G is symmetric.

(ii) \Rightarrow (i). By Proposition 3, the map $q \rightarrow q^*$ is an automorphism of $\psi(Con_{n+1})$. This map transforms a facet G with facet vector g into a facet G^* with facet vector g^* . If G is symmetric, then $G^* = G$, and therefore $g^* = g$.

(iii) \Rightarrow (i). Let $f = f^V + f^0$ be a facet vector of a facet F of Con_{n+1} such that $f^0 = 0$. Then the facet F is zero-lifting of the facet $F^V = F \cap \mathbb{R}^E$ of the cone Con_n . In this case, f^V is also a facet vector of the facet $G = \psi(F)$ of $\psi(Con_{n+1})$. Obviously, $(f^V)^* = f^V$.

(iii) \Rightarrow (iv). This implication is implied by definition of zero-lifting of a facet of the cone $\psi(Con_{n+1})$.

(iv) \Rightarrow (i). The map χ induces a bijection between $\pi(F)$ and $\psi(F)$. Since $\pi(F)$ is zero-lifting of F^V , the facet vector of $\pi(F)$ belongs to \mathbb{R}^E . This implies that the facet vector g of $\psi(F)$ belongs to $\mathbb{R}_s^{E^O}$, i.e. $g^* = g$. \square

The symmetry group of Con_{n+1} is the symmetric group Σ_{n+1} of permutations of indices (see [DL97]). The group Σ_n is a subgroup of the symmetry group of the cone $\psi(Con_{n+1})$. The full symmetry group of $\psi(Con_{n+1})$ is $\Sigma_n \times \Sigma_2$, where Σ_2 corresponds to the map $q \rightarrow q^*$ for $q \in \psi(Con_{n+1})$. By Proposition 4, the set of facets of $\psi(Con_{n+1})$ is partitioned into pairs G, G^* . But it turns out that there are pairs such that $G^* = \sigma(G)$, where $\sigma \in \Sigma_n$.

9 Projections of hypermetric facets

The metric cone Met_{n+1} , the hypermetric cone Hyp_{n+1} and the cut cone Cut_{n+1} lying in the space $\mathbb{R}^{E \cup E_0}$ have an important class of *hypermetric* facets, that contains the class of *triangular* facets.

Let $b_i, i \in V$, be integers such that $\sum_{i \in V} b_i = \mu$, where $\mu = 0$ or $\mu = 1$. Usually these integers are denoted as a sequence (b_1, b_2, \dots, b_n) , where $b_i \geq b_{i+1}$. If, for some i , we have $b_i = b_{i+1} = \dots = b_{i+m-1}$, then the sequence is shortened as $(b_1, \dots, b_i^m, b_{i+m}, \dots, b_n)$.

One relates to this sequence the following inequality of type $b = (b_1, \dots, b_n)$

$$(f(b), x) = - \sum_{i,j \in V} b_i b_j x_{(ij)} \geq 0,$$

where $x = \{x_{(ij)}\} \in \mathbb{R}^E$ and the vector $f(b) \in \mathbb{R}^E$ has coordinates $f(b)_{(ij)} = -b_i b_j$. This inequality is called of *negative* or *hypermetric* type if in the sum $\sum_{i \in V} b_i = \mu$ we have $\mu = 0$ or $\mu = 1$, respectively.

The set of hypermetric inequalities on the set $V \cup \{0\}$ determines a hypermetric cone Hyp_{n+1} . There are infinitely many hypermetric inequalities for metrics on $V \cup \{0\}$. But it is proved in [DL97], that only finite number of these inequalities determines facets of Hyp_{n+1} . Since triangle inequalities are inequalities $(f(b), x) \geq 0$ of type $b = (1^2, 0^{n-3}, -1)$, the hypermetric cone Hyp_{n+1} is contained in Met_{n+1} , i.e. $Hyp_{n+1} \subseteq Met_{n+1}$ with equality for $n = 2$.

The hypermetric inequality $(f(b), x) \geq 0$ takes the following form on the set $V \cup \{0\}$.

$$- \sum_{i,j \in V \cup \{0\}} b_i b_j x_{(ij)} = - \sum_{(ij) \in E} b_i b_j x_{(ij)} - \sum_{i \in V} b_0 b_i x_{(0i)} \geq 0. \quad (14)$$

If we decompose the vector $f(b)$ as $f(b) = f^V(b) + f^0(b)$, then $f^V(b)_{(ij)} = -b_i b_j, (ij) \in E$, and $f^0(b)_{(0i)} = -b_0 b_i, i \in V$.

Let, for $S \subseteq V$, the equality $\sum_{i \in S} b_i = 0$ hold. Denote by b^S a sequence such that $b_i^S = -b_i$ if $i \in S$ and $b_i^S = b_i$ if $i \notin S$. The sequence b^S is called *switching* of b by the set S .

The hypermetric cone Hyp_{n+1} has the following property (see [DL97]). If an inequality $(f(b), x) \geq 0$ defines a facet and $\sum_{i \in S} b_i = 0$ for some $S \subseteq V \cup \{0\}$, then the inequality $(f(b^S), x) \geq 0$ defines a facet, too.

Proof of Proposition 2 for Hyp_{n+1} .

Consider the inequality (14), where $(f^0(b), e_0) = -\sum_{i \in V} b_0 b_i = 0$. Then $\sum_{i \in V} b_i = 0$. Hence the cone Hyp_{n+1} has similar inequality for b^V , where $b_i^V = -b_i$ for all $i \in V$. Hence if one of these inequalities defines a facet so does another. Obviously, $f^0(b^V) = -f^0(b)$. Hence these facets satisfy the assertion of Proposition 2. \square

Theorem 3. Let $(f(b), x) \geq 0$ define a hypermetric facet of a cone in the space $\mathbb{R}^{E \cup E_0}$. Then the map ψ transforms it either in a hypermetric facet if $b_0 = 0$ or in a distortion of a facet of negative type if $b_0 = 1$. Otherwise, the projection is not a facet.

Proof. By Section 8, the map ψ transforms the hypermetric inequality (14) for $x \in \mathbb{R}^{E \cup E_0}$ into the following inequality

$$- \sum_{(ij) \in E} b_i b_j q_{(ij)} - b_0 \sum_{i \in V} b_i w_i \geq 0$$

for $q = \sum_{(ij) \in E} q_{(ij)} \varphi(e_{(ij)}) + \sum_{i \in V} w_i q(i) \in Q_n$.

Since $f(b)$ determines a hypermetric inequality, we have $b_0 = 1 - \sum_{i \in V} b_i = 1 - \mu$. So, the above inequality takes the form

$$\sum_{(ij) \in E} b_i b_j q_{(ij)} \leq (\mu - 1) \sum_{i \in V} b_i w_i.$$

By Theorem 1, this facet is projected by the map ψ into a facet if and only if $(f(b), e_0) = 0$, where $e_0 = \sum_{i \in V} e_{(0i)}$. We have

$$(f(b), e_0) = \sum_{i \in V} f(b)_{(0i)} = - \sum_{i \in V} b_0 b_i = -b_0 \mu = (\mu - 1) \mu.$$

This implies that the hypermetric facet-defining inequality $(f(b), x) \geq 0$ is transformed into a facet-defining inequality if and only if either $\mu = 0$ and then $b_0 = 1$ or $\mu = 1$ and then $b_0 = 0$. So, we have

if $\mu = 1$ and $b_0 = 0$, then the above inequality is a usual hypermetric inequality in the space $\psi(\mathbb{R}^E) = \varphi(\mathbb{R}^E) = \mathbb{R}_s^{E^O}$;

if $\mu = 0$ and $b_0 = 1$, then the above inequality is the following distortion of an inequality of negative type

$$- \sum_{(ij) \in E} b_i b_j q_{(ij)} - \sum_{i \in V} b_i w_i \geq 0, \text{ where } \sum_{i \in V} b_i = 0. \quad (15)$$

□

Comparing (7) with the inequality (15), we see that a canonical facet vector $g(b)$ of a facet of $\psi(Hyp_{n+1})$ has the form $g(b) = g^s(b) + g^a(b)$, where $g_{ij}(b) = g_{(ij)}(b) + v_i - v_j$, and

$$g_{(ij)}(b) = -b_i b_j, \quad v_i = -\frac{1}{n} b_i \text{ for all } i \in V.$$

Define a cone of weighted quasi-hyper-metrics $WQHyp_n = \psi(Hyp_{n+1})$. We can apply Proposition 3, in order to obtain the following assertion.

Proposition 6. *The map $q \rightarrow q^*$ preserves the cone $WQHyp_n$, i.e.*

$$(WQHyp_n)^* = WQHyp_n.$$

In other words, if $q \in WQHyp_n$ has weights $w_i, i \in V$, then the cone $WQHyp_n$ has a point q^ with weights $-w_i, i \in V$.* □

10 Generalizations of metrics

The metric cone Met_{n+1} is defined in the space $\mathbb{R}^{E \cup E_0}$. It has an extreme ray which is spanned by the vector $e_0 = \sum_{i \in V} e_{(0i)} \in \mathbb{R}^{E_0}$. Facets of Met_{n+1} are defined by the following set of triangle inequalities, where $d \in Met_{n+1}$.

Triangle inequalities of the sub-cone Met_n that define facets of Met_{n+1} that are zero-lifting and contain e_0 :

$$d_{(ik)} + d_{(kj)} - d_{(ij)} \geq 0, \text{ for } i, j, k \in V. \quad (16)$$

Triangle inequalities defining facets that are not zero-lifting and contain the extreme ray l_0 spanned by the vector e_0 :

$$d_{(ij)} + d_{(j0)} - d_{(i0)} \geq 0 \text{ and } d_{(ij)} + d_{(i0)} - d_{(j0)} \geq 0, \text{ for } i, j \in V. \quad (17)$$

Triangle inequalities defining facets that do not contain the extreme ray l_0 and do not define facets of Met_n .

$$d_{(i0)} + d_{(j0)} - d_{(ij)} \geq 0, \text{ for } i, j \in V. \quad (18)$$

One can say that the cone $Met_n \in \mathbb{R}^{E_0}$ is lifted into the space $\mathbb{R}^{E \cup E_0}$ using restrictions (17) and (18). Note that the inequalities (17) and (18) imply the following inequalities of non-negativity

$$d_{(i0)} \geq 0, \text{ for } i \in V. \quad (19)$$

A cone defined by inequalities (16) and (19) is called by cone $WMet_n$ of *weighted metrics* (d, w) , where $d \in Met_n$ and $w_i = d_{(0i)}$ for $i \in V$ are *weights*.

If weights $w_i = d_{(0i)}$ satisfy also the inequalities (17) additionally to the inequalities (19), then the weighted metrics (d, w) form a cone $dWMet_n$ of *down-weighted* metrics. If metrics have weights that satisfy the inequalities (19) and (18), then these metrics are called *up-weighted* metrics. Detail see in [DD10], [DDV11].

Above defined generalizations of metrics are functions on unordered pairs $(ij) \in E \cup E_0$. Generalizations of metrics as functions on ordered pairs $ij \in E^\circ$ are called *quasi-metrics*.

The cone $QMet_n$ of quasi-metrics is defined in the space \mathbb{R}^{E° by non-negativity inequalities $q_{ij} \geq 0$ for all $ij \in E^\circ$, and by triangle inequalities $q_{ij} + q_{jk} - q_{ik} \geq 0$ for all ordered triples ijk for each $q \in QMet_n$. Below we consider in $QMet_n$ a sub-cone $WQMet_n$ of weighted quasi-metrics.

11 Cone of weighted quasi-metrics

We call a quasi-metric q *weighted* if it belongs to the subspace $Q_n \subset \mathbb{R}^{E^\circ}$. So, we define

$$WQMet_n = QMet_n \cap Q_n.$$

A *quasi-metric* q is called *weightable* if there are weights $w_i \geq 0$ for all $i \in V$ such that the following equalities hold

$$q_{ij} + w_i = q_{ji} + w_j$$

for all $i, j \in V, i \neq j$. Since $q_{ij} = q_{ij}^s + q_{ij}^a$, we have $q_{ij} + w_i = q_{ij}^s + q_{ij}^a + w_i = q_{ji}^s + q_{ji}^a + w_j$, i.e. $q_{ij}^a - q_{ji}^a = 2q_{ij}^a = w_j - w_i$, what means that, up to multiple $\frac{1}{2}$ and sign, the antisymmetric part of q_{ij} is $w_i - w_j$. So, weightable quasi-metrics are weighted.

Note that weights of a weighted quasi-metric are defined up to an additive constant. So, if we take weights non-positive, we obtain a weightable quasi-metric. Hence, sets of weightable and weighted quasi-metrics coincide.

By definition of the cone $WQMet_n$ and by symmetry of this cone, the triangle inequality $q_{ij} + q_{jk} - q_{ik} \geq 0$ and non-negativity inequality $q_{ij} \geq 0$ determine facets of the cone $WQMet_n$. Facet vectors of these facets are

$$t_{ijk} = e_{ij} + e_{jk} - e_{ik} \text{ and } e_{ij},$$

respectively. It is not difficult to verify that $t_{ijk}, e_{ij} \notin Q_n$. Hence these facet vectors are not canonical. Below, we give canonical representatives of these facet vectors.

Let $T(ijk) \subseteq K_n^\circ$ be a triangle of K_n° with direct arcs ij, jk, ki and opposite arcs ji, kj, ik . Hence

$$f^{T(ijk)} = (e_{ij} + e_{jk} + e_{ki}) - (e_{ji} + e_{kj} + e_{ik}).$$

Proposition 7. *Canonical representatives of facet vectors t_{ijk} and e_{ij} are*

$$t_{ijk} + t_{ijk}^* = t_{ijk} + t_{kji}, \text{ and } g(ij) = (e_{ij} + e_{ji}) + \frac{1}{n}(p(i) - p(j)),$$

respectively.

Proof. We have $t_{ijk} - f^{T(ijk)} = e_{ji} + e_{kj} - e_{ki} = t_{kji} = t_{ijk}^*$. This implies that the facet vectors t_{ijk} and t_{kji} determine the same facet, and the vector $t_{ijk} + t_{kji} \in \mathbb{R}_s^{E^\circ}$ is a canonical representative of facet vectors of this facet. We obtain the first assertion of Proposition.

Consider now the facet vector e_{ij} . It is more convenient to take the doubled vector $2e_{ij}$. We show that the vector

$$g(ij) = 2e_{ij} - \frac{1}{n} \sum_{k \in V - \{i, j\}} f^{T(ijk)}.$$

is a canonical representative of the facet vector $2e_{ij}$. It is sufficient to show that $g(ij) \in Q_n$, i.e. $g_{kl}(ij) = g_{kl}^s(ij) + w_k - w_l$. In fact, we have $g_{ij}(ij) = 2 - \frac{n-2}{n} = 1 + \frac{2}{n}$, $g_{ji}(ij) = \frac{n-2}{n} = 1 - \frac{2}{n}$, $g_{ik}(ij) = -g_{ki}(ij) = \frac{1}{n}$, $g_{jk}(ij) = -g_{kj}(ij) = -\frac{1}{n}$, $g_{kk'}(ij) = 0$. Hence we have

$$g^s(ij) = e_{ij} + e_{ji}, w_i = -w_j = \frac{1}{n}, \text{ and } w_k = 0 \text{ for all } k \in V - \{i, j\}.$$

These equalities imply the second assertion of Proposition. \square

Let τ_{ijk} be a facet vector of a facet of Met_n determined by the inequality $d_{(ij)} + d_{(jk)} - d_{(ik)} \geq 0$. Then $t_{ijk} + t_{kji} = \varphi(\tau_{ijk})$, where the map $\varphi : \mathbb{R}^E \rightarrow \mathbb{R}_s^{E^O}$ is defined in Section 2. Obviously, a triangular facet is symmetric.

Recall that $q_{ij} = q_{(ij)} + w_i - w_j$ if $q \in WQMet_n$. Let $i, j, k \in V$. It is not difficult to verify that the following equalities hold:

$$q_{ij}^s + q_{jk}^s - q_{ik}^s = q_{ij} + q_{jk} - q_{ij} \geq 0. \quad (20)$$

Since $q_{ij}^s = q_{ji}^s = q_{(ij)}$, these inequalities show that the symmetric part q^s of the vector $q \in WQMet_n$ is a semi-metric. Hence if $w_i = w$ for all $i \in V$, then the quasi-semi-metric $q = q^s$ itself is a semi-metric. This implies that the cone $WQMet_n$ contains the semi-metric cone Met_n . Moreover, $Met_n = WQMet_n \cap \mathbb{R}_s^{E^O}$.

Now we show explicitly how the map ψ transforms the cones Met_{n+1} and $dWMet_n$ into the cone $WQMet_n$.

Theorem 4. *The following equalities hold*

$$\psi(Met_{n+1}) = \psi(dWMet_n) = WQMet_n \text{ and } WQMet_n^* = WQMet_n.$$

Proof. All facets of the metric cone Met_{n+1} of metrics on the set $V \cup \{0\}$ are given by triangular inequalities $d_{(ij)} + d_{(ik)} - d_{(kj)} \geq 0$. They are hypermetric inequalities $(g(b), d) \geq 0$, where b has only three non-zero values $b_j = b_k = 1$ and $b_i = -1$ for some triple $\{ijk\} \subseteq V \cup \{0\}$. By Theorem 3, the map ψ transforms this facet into a hypermetric facet, i.e. into a triangular facets of the cone $\psi(Met_{n+1})$ if and only if $b_0 = 0$, i.e. if $0 \notin \{ijk\}$. If $0 \in \{ijk\}$, then, by the same theorem, the equality $b_0 = 1$ should be satisfied. This implies $0 \in \{jk\}$. In this case the facet defining inequality has the form (15), that in the case $k = 0$, is

$$q_{(ij)} + w_i - w_j \geq 0.$$

This inequality is the non-negativity inequality $q_{ij} \geq 0$.

If $b_i = 1, b_j = -1$ and $k = 0$, the inequality $d_{(ij)} + d_{j0} - d_{(0i)} \geq 0$ is transformed into inequality

$$q_{(ij)} + w_j - w_i \geq 0, \text{ i.e. } q_{ij}^* \geq 0.$$

This inequality and inequalities (20) imply the last equality of this Theorem.

The inequalities (18) define facets F of Met_{n+1} and $dWMet_n$ that do not contain the extreme ray l_0 . Hence, by Theorem 3, $\psi(F)$ are not facets of $WQMet_n$. But, recall that the cone $dWMet_n$ contains all facet of Met_{n+1} excluding facets defined by the inequalities (18). Instead of these facets, the cone $dWMet_n$ has facets G_i defined by the non-negativity equalities (19) with facet vectors $e_{(0i)}$ for all $i \in V$. Obviously all these facets do not contain the extreme ray l_0 . Hence, by Theorem 2, $\psi(G_i)$ is not a facet of $\psi(dWMet_n)$. Hence we have also the equality $WQMet_n = \psi(dWMet_n)$. \square

Remark. Facet vectors of facets of Met_{n+1} that contain the extreme ray l_0 spanned by the vector e_0 are $\tau_{ijk} = \tau_{ijk}^V, \tau_{ij0} = \tau^V + \tau^0$ and $\tau_{j0i} = \tau^V - \tau^0$, where $\tau^V = e_{(ij)}$ and $\tau^0 = e_{(j0)} - e_{(i0)}$. Hence Proposition 2 is true for Met_{n+1} , and we can apply Proposition 3 in order to obtain the equality $WQMet_n^* = WQMet_n$ of Theorem 4.

12 The cone Cut_{n+1}

The cut vectors $\delta(S) \in \mathbb{R}^{E \cup E_0}$ for all $S \subseteq V \cup \{0\}$ span all extreme rays of the cut cone $Cut_{n+1} \subset \mathbb{R}^{E \cup E_0}$. In

other words, Cut_{n+1} is the conic hull of all cut vectors. Since the cone Cut_{n+1} is full dimensional, its dimension is dimension of the space $\mathbb{R}^{E \cup E_0}$ that is $\frac{n(n+1)}{2}$.

Recall that $\delta(S) = \delta(V \cup \{0\} - S)$. Hence we can consider only S such that $S \subseteq V$, i.e. $0 \notin S$. Moreover, by Section 5,

$$\delta(S) = \sum_{i \in S, j \notin S} e_{(ij)} = \sum_{i \in S, j \in V-S} e_{(ij)} + \sum_{i \in S} e_{(0i)} = \delta^V(S) + \sum_{i \in S} e_{(0i)}, \quad (21)$$

where $\delta^V(S)$ is restriction of $\delta(S)$ on the space $\mathbb{R}^E = \psi(\mathbb{R}^E)$. Note that

$$\delta(V) = \delta(\{0\}) = \sum_{i \in V} e_{(0i)} = e_0.$$

Consider a facet F of Cut_{n+1} . Let f be facet vector of F . Set

$$R(F) = \{S \subseteq V : (f, \delta(S)) = 0\}.$$

For $S \in R(F)$, the vector $\delta(S)$ is called *root* of the facet F . By (21), for $S \in R(F)$, we have

$$(f, \delta(S)) = (f, \delta^V(S)) + \sum_{i \in S} f_{(0i)} = 0. \quad (22)$$

We represent each facet vector of Cut_{n+1} as $f = f^V + f^0$, where $f^V \in \mathbb{R}^E$ and $f^0 \in \mathbb{R}^{E_0}$.

The set of facets of the cone Cut_{n+1} is partitioned onto equivalence classes by *switchings* (see [DL97]). For each $S, T \subset V \cup \{0\}$, the switching by the set T transforms the cut vector $\delta(S)$ into the vector $\delta(S \Delta T)$, where Δ is symmetric difference, i.e. $S \Delta T = S \cup T - S \cap T$. It is proved in [DL97] that if $T \in R(F)$, then $\{\delta(S \Delta T) : S \in R(F)\}$ is the set of roots of the switched facet $F^{\delta(T)}$ of Cut_{n+1} . Hence $R(F^{\delta(T)}) = \{S \Delta T : S \in R(F)\}$.

Let F be a facet of Cut_{n+1} . Then F contains the vector $e_0 = \delta(V)$ if and only if $V \in R(F)$. Hence Lemma 1 below is an extended reformulation of Proposition 2.

Lemma 1. *Let F be a facet of Cut_{n+1} such that $V \in R(F)$. Let $f = f^V + f^0$ be facet vector of F . Then the vector $f^* = f^V - f^0$ is facet vector of switching $F^{\delta(V)}$ of the facet F , and $V \in R(F^{\delta(V)})$.*

Proof. Since $V \in R(F)$, $F^{\delta(V)}$ is a facet of Cut_{n+1} . Since $S \Delta V = V - S = \bar{S}$, for $S \subseteq V$, we have

$$R(F^{\delta(V)}) = \{\bar{S} : S \in R(F)\}.$$

Since $\emptyset \in R(F)$, the set $\emptyset \Delta V = V \in R(F^{\delta(V)})$. Now, using (22), for $S \in R(F^{\delta(V)})$, we have

$$(f^*, \delta(S)) = ((f^V - f^0), \delta(S)) = (f^V, \delta^V(S)) - \sum_{i \in S} f_{(0i)}.$$

Note that $\delta^V(\bar{S}) = \delta^V(S)$, and, since $V \in R(F)$, $\delta(V) = \delta(\{0\})$, we have $(f, \delta(V)) = \sum_{i \in V} f_{(0i)} = 0$. Hence $\sum_{i \in \bar{S}} f_{(0i)} = -\sum_{i \in S} f_{(0i)}$. It is easy to see, that $(f^*, \delta(S)) = (f, \delta(\bar{S}))$. Since $S \in R(F^{\delta(V)})$ if and only if $\bar{S} \in R(F)$, we see that f^* is a facet vector of $F^{\delta(V)}$. \square

The set of facets of Cut_{n+1} is partitioned into orbits under action of the permutation group Σ_{n+1} . But some permutation non-equivalent facets are equivalent under switchings. We say that two facets F, F' of Cut_{n+1} belong to the same *type* if there are $\sigma \in \Sigma_{n+1}$ and $T \subseteq V$ such that $\sigma(F') = F^{\delta(T)}$.

13 Cone $OCut_n$

Denote by $OCut_n \subset \mathbb{R}^{E^O}$ the cone whose extreme rays are spanned by ocut vectors $c(S)$ for all $S \subset V$, $S \neq \emptyset, V$. In other words, let

$$OCut_n = \{c \in Q_n : c = \sum_{S \subset V} \alpha_S c(S), \alpha_S \geq 0\}.$$

Coordinates c_{ij} of a vector $c \in OCut_n$ are given in (6), where $w_i \geq 0$ for all $i \in V$. Hence $OCut_n \subset Q_n$. Recall that

$$c(S) = \frac{1}{2}(\delta^{\mathcal{O}}(S) + \sum_{i \in S} p(i)), \quad (23)$$

where $\delta^{\mathcal{O}}(S) = \varphi(\delta^V(S))$. Note that $\delta^{\mathcal{O}}(\overline{S}) = \delta^{\mathcal{O}}(S)$ and $p(\overline{S}) = -p(S)$, where $\overline{S} = V - S$.

Denote by $Cut_n^{\mathcal{O}} = \varphi(Cut_n)$ the cone generated by $\delta^{\mathcal{O}}(S)$ for all $S \subset V$. The vectors $\delta^{\mathcal{O}}(S)$ for all $S \subset V$, $S \neq \emptyset, V$, are all extreme rays of the cone $Cut_n^{\mathcal{O}}$ that we identify with Cut_n embedded into the space $\mathbb{R}^{E^{\mathcal{O}}}$.

Lemma 2. For $S \subseteq V$, the following equality holds

$$\psi(\delta(S)) = 2c(S).$$

Proof. According to Section 8, $\psi(\delta^V(S)) = \varphi(\delta^V(S)) = \delta^{\mathcal{O}}(S)$. Besides, $\psi(e_{(0i)}) = p(i)$ for all $i \in V$. Hence, using (21), we obtain

$$\psi(\delta(S)) = \psi(\delta^V(S)) + \sum_{i \in S} \psi(e_{(0i)}) = \varphi(\delta^V(S)) + \sum_{i \in S} p(i) = \delta^{\mathcal{O}}(S) + p(S).$$

Recall that $\psi(\delta(V)) = \psi(e_0) = \mathbf{0}$ and $c(V) = 0$. Hence, according to (23), we obtain

$$\psi(\delta(S)) = 2c(S), \text{ for all } S \subseteq V.$$

Lemma is proved. □

Theorem 5. The following equalities hold

$$\psi(Cut_{n+1}) = OCut_n \text{ and } OCut_n^* = OCut_n.$$

Proof. Recall that the conic hull of vectors $\delta(S)$ for all $S \subseteq V$ is Cut_{n+1} . The conic hull of vectors $c(S)$ for all $S \subset V$ is the cone $OCut_n$. Since $\psi(\delta(V)) = c(V) = \mathbf{0}$, the first result follows.

The equality $OCut_n^* = OCut_n$ is implied by the equalities $c^*(S) = c(\overline{S})$ for all $S \subseteq V$.

By Lemma 1, the equality $OCut_n^* = OCut_n$ is a special case $Con_{n+1} = Cut_{n+1}$ of Proposition 3. □

14 Facets of $OCut_n$

Lemma 3. Let F be a facet of Cut_{n+1} . Then $\psi(F)$ is a facet of $OCut_n$ if and only if $V \in R(F)$.

Proof. By Theorem 2, $\psi(F)$ is a facet of $OCut_n$ if and only if $e_0 = \delta(V) \subset F$, i.e. if and only if $V \in R(F)$. □

For a facet G of $OCut_n$ with facet vector g , we set

$$R(G) = \{S \subseteq V : (g, c(S)) = 0\}$$

and call the vector $c(S)$ for $S \in R(G)$ by *root* of the facet G .

Note that $\delta(\emptyset) = \mathbf{0}$ and $c(\emptyset) = c(V) = \mathbf{0}$. Hence $\emptyset \in R(F)$ and $\emptyset \in R(G)$ for all facet F of Cut_{n+1} and all facets G of $OCut_n$. The roots $\delta(\emptyset) = \mathbf{0}$ and $c(\emptyset) = c(V) = \mathbf{0}$ are called *trivial roots*.

Proposition 8. For a facet F of Cut_{n+1} , let $G = \psi(F)$ be a facet of $OCut_n$. Then the following equality holds

$$R(G) = R(F).$$

Remark. We give two proofs of this equality. Both are useful.

First proof. According to Section 8, the map ψ transforms an inequality $(f, x) \geq 0$ defining a facet of Cut_{n+1} into the inequality (12) defining the facet $G = \psi(F)$ of $OCut_n$. Recall the the inequality (12) relates to the representation of vectors $q \in Q_n$ in the basis $\{\varphi(e_{ij}), p(i)\}$, i.e. $q = \sum_{(ij) \in E} q_{(ij)} \varphi(e_{ij}) + \sum_{i \in V} w_i p(i)$. Let $q = c(S)$ for $S \in R(G)$. Then, according to (23), we have $q_{(ij)} = \frac{1}{2} \delta_{(ij)}^V(S)$, $w_i = \frac{1}{2}$ for $i \in S$ and $w_i = 0$ for $i \in \bar{S}$. Hence, omitting the multiple $\frac{1}{2}$, the inequality in (12) gives the following equality

$$\sum_{(ij) \in E} f_{(ij)} \delta_{(ij)}^V(S) + \sum_{i \in S} f_{(0i)} = 0$$

which coincides with (22). This implies the assertion of this Proposition.

Second proof. By Theorem 2, $\psi(l)$ is an extreme ray of $\psi(F)$ if and only if l is an extreme ray of F and $l \neq l_0$. Since l is spanned by $\delta(S)$ for some $S \in R(F)$ and $\psi(l)$ is spanned by $\psi(\delta(S)) = c(S)$, we have $R(G) = \{S \subset V : S \in R(F)\}$. Since $c(V) = \mathbf{0}$, we can suppose that $V \in R(G)$, and then $R(G) = R(F)$. \square

Remark. Note that $\delta(V) = \delta(\{0\}) = e_0 \neq \mathbf{0}$ is a non-trivial root of F , i.e. $V \in R(F)$. But $c(V) = \psi(\delta(V)) = \mathbf{0}$ is a trivial root of $R(G)$.

Recall that, for a subset $T \subseteq V$, we set $\bar{T} = V - T$. Note that $\bar{\bar{T}} = T$ and $\bar{T} \neq V \cup \{0\} - T$.

Lemma 4. *Let F be a facet of Cut_{n+1} , and $T \in R(F)$. Then the image $\psi(F^{\delta(T)})$ of the switched facet $F^{\delta(T)}$ is a facet of $OCut_n$ if and only if $\bar{T} \in R(F)$.*

Proof. By Lemma 3, $\psi(F^{\delta(T)})$ is a facet of $OCut_n$ if and only if $V \in R(F^{\delta(T)})$, i.e. if and only if $V \Delta T = \bar{T} \in R(F)$. \square

For a facet G of $OCut_n$, define $G^{\delta(T)}$ as the conic hull of $c(S \Delta T)$ for all $S \in R(G)$. Since each facet G of $OCut_n$ is $\psi(F)$ for some facet F of Cut_{n+1} , Lemma 4 and Proposition 8 imply the following assertion.

Theorem 6. *Let G be a facet of $OCut_n$. Then $G^{\delta(T)}$ is a facet of $OCut_n$ if and only if $T, \bar{T} \in R(G)$, and then $R(G^{\delta(T)}) = \{S \Delta T : S \in R(G)\}$.* \square

Theorem 6 asserts that the set of facets of the cone $OCut_n$ is partitioned onto equivalence classes by switchings $G \rightarrow G^{\delta(T)}$, where $T, \bar{T} \in R(G)$.

The case $T = V$ in Theorem 6 plays a special role. Recall that $V \in R(F)$ if F is a facet of Cut_{n+1} such that $\psi(F)$ is a facet of $OCut_n$. Hence Lemma 1 and Proposition 3 imply the following fact.

Proposition 9. *Let F be a facet of Cut_{n+1} such that $\psi(F)$ is a facet of $OCut_n$. Let $g = g^s + g^a$ be a facet vector of the facet $\psi(F)$. Then the vector $g^* = g^s - g^a$ is a facet vector of the facet $\psi(F^{\delta(V)}) = (\psi(F))^* = (\psi(F))^{\delta(V)}$ such that $R((\psi(F))^*) = \{\bar{S} : S \in R(F)\}$.* \square

Recall that roughly speaking $OCut_n$ is projection of Cut_{n+1} along the vector $\delta(V) = \delta(\{0\})$.

Let $\sigma \in \Sigma_n$ be a permutation of the set V . For a vector $q \in \mathbb{R}^{E \cup V}$, we have $\sigma(q)_{ij} = q_{\sigma(i)\sigma(j)}$. Obviously if g is a facet vector of a facet G of $OCut_n$, then $\sigma(g)$ is the facet vector of the facet $\sigma(G) = \{\sigma(q) : q \in G\}$.

Note that, by Proposition 9, the switching by V is equivalent to the operation $q \rightarrow q^*$. Hence the symmetry group of $OCut_n$ contains the group $\Sigma_n \times \Sigma_2$, where Σ_2 relates to the map $q \rightarrow q^*$ for $q \in OCut_n$.

Theorem 7. *The group $\Sigma_n \times \Sigma_2$ is the symmetry group of the cone $OCut_n$.*

Proof. Let γ be a symmetry of $OCut_n$. Then γ is a symmetry of the set $\mathcal{F}(e_0)$ of facets F of the cone Cut_{n+1} containing the vector e_0 . The symmetry group $\Gamma(e_0)$ of the set $\mathcal{F}(e_0)$ is a subgroup of the symmetry group of the cut-polytope Cut_{n+1}^\square . In fact, $\Gamma(e_0)$ is stabilizer of the edge e_0 of the polytope Cut_{n+1}^\square . But it is well-known that

$\Gamma(e_0)$ consists of the switching by V and permutations $\sigma \in \Sigma_{n+1}$ leaving the edge e_0 non-changed. The map ψ transforms these symmetries of $\mathcal{F}(e_0)$ into symmetries $\sigma \in \Sigma_n$ and $q \rightarrow q^*$ of the cone $OCut_n$. \square

The set of all facets of $OCut_n$ is partitioned onto orbits of facets that are equivalent by the symmetry group $\Sigma_n \times \Sigma_2$. It turns out that, for some facets G , subsets $S \in R(G)$ and permutations $\sigma \in \Sigma_n$, we have $G^{\delta(S)} = \sigma(G)$.

By Proposition 5, if a facet of Cut_{n+1} is zero-lifting of a facet F^V of Cut_n , then the facet $G = \psi(F)$ of $OCut_n$ is symmetric and $G = G^* = G^{\delta(V)}$ is zero-lifting of F^V .

So, there are two important classes of orbits of facets of $OCut_n$. Namely, the orbits of symmetric facets, that are zero-lifting of facets of Cut_n , and orbits of asymmetric facets that are ψ -images of facets of Cut_{n+1} and are not zero-lifting.

15 Cases $3 \leq n \leq 6$

It is worth to compare results of this Section with Table 2 of [DDV11].

Most of described below facets are hypermetric or negative type. We give here the corresponding vectors b in accordance with Section 9.

n=3. Note that $Cut_4 = Hyp_4 = Met_4$. Hence

$$OCut_3 = WQHyp_3 = WQMet_3.$$

All these cones have two orbits of facets: one orbit of non-negativity facets with $b = (1, 0, -1)$ and another orbit of triangular facets with $b = (1^2, -1)$.

n=4. We have $Cut_5 = Hyp_5 \subset Met_5$. Hence

$$OCut_4 = WQHyp_4 \subset WQMet_4.$$

The cones $Hyp_5 = Cut_5$ have two orbits of facets: triangular and pentagonal facets. Recall that a triangular facet with facet vector τ_{ijk} is zero-lifting if $0 \notin \{ijk\}$. Hence the cones $WQHyp_4 = OCut_4$ have three orbits of facets: of non-negativity with $b = (1, 0^2, -1)$, triangular with $b = (1^2, 0, -1)$ and weighted version of negative type with $b = (1^2, -1^2)$.

n=5. We have again $Cut_6 = Hyp_6 \subset Met_6$. Hence

$$OCut_5 = WQHyp_5 \subset WQMet_5.$$

The cones $Hyp_6 = Cut_6$ have four orbits of facets, all are hypermetric: triangular with $b = (1^2, 0^3, -1)$, pentagonal with $b = (1^3, 0, -1^2)$ and two more types, one with $b = (2, 1^2, -1^3)$ and its switching with $b = (1^4, -1, -2)$. These four types provide 6 orbits of facets of the cones $WQHyp_5 = OCut_5$: non-negativity with $b = (1, 0^3, -1)$, triangular with $b = (1^2, 0^2, -1)$, of negative type with $b = (1^2, 0, -1^2)$, pentagonal with $b = (1^3, -1^2)$, and two of negative type with $b = (2, 1, -1^3)$ and $b = (1^3, -1, -2)$.

The last two types belong to the same orbit of the full symmetry group $\Sigma_5 \times \Sigma_2$. Hence the cone $OCut_5$ has 5 orbits of facets under action of its symmetry group.

n=6. Now, we have $Cut_7 \subset Hyp_7 \subset Met_7$. Hence

$$OCut_6 \subset WQHyp_6 \subset WQMet_6.$$

The cone Cut_7 has 36 orbits of facets under action of the permutation group Σ_7 . Switchings contract these orbits into 11 types F_k , $1 \leq k \leq 11$, (see [DL97], Sect. 30.6). J.Vidali compute orbits of facets of $OCut_6$ under action of the group Σ_6 . Using these computations, we give in Table below numbers of orbits of facets of cones Cut_7 and $OCut_6$ (cf. Figure 30.6.1 of [DL97]).

The first row of Table gives types of facets of Cut_7 . In the second row of Table, for each type F_k , numbers of orbits of facets of Cut_7 of type F_k under action of the group Σ_7 . The third row of Table, for each type F_k , gives numbers of orbits of facets of $OCut_6$ that are obtained from facets of type F_k under action of the group Σ_6 . The fourth row gives, for each type F_k , numbers of orbits of facets of $OCut_6$ that are obtained from facets of type F_k under action of the group $\Sigma_6 \times \Sigma_2$.

The last column of Table gives total numbers of orbits of facets of the cones Cut_7 and $OCut_6$.

Table.

types	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	$ \Omega $
Σ_7	1	1	2	1	3	2	4	7	5	3	7	36
Σ_6	2	2	4	1	3	2	7	13	6	6	15	61
$\Sigma_6 \times \Sigma_2$	2	2	3	1	2	1	4	7	3	4	8	37

The first three types F_1, F_2, F_3 relate to 4 orbits of hypermetric facets $F(b)$ of Cut_7 that are zero-lifting, where $b = (1^2, 0^4, -1)$, $b = (1^3, 0^2, -1^2)$ and $b = (2, 1^2, 0, -1^3)$, $b = (1^4, 0, -1, -2)$. Each of these four orbits of facets of Cut_7 under action of Σ_7 gives two orbits of facets of $OCut_6$ under action of the group Σ_6 .

The second three types F_4, F_5, F_6 relate to 6 orbits of hypermetric facets $F(b)$ of Cut_7 that are not zero-lifting. Each of these 6 orbits gives one orbit of facets of $OCut_6$ under action of the group Σ_6 .

The third three types F_7, F_8, F_9 relate to 16 orbits of facets of clique-web types $CW_1^7(b)$. These 16 orbits give 26 orbits of facets of $OCut_6$ under action of Σ_6 .

The last two types $F_{10} = Par_7$ and Gr_7 are special (see [DL97]). They relate to 10 orbits of Cut_7 , that give 21 orbits of facets of $OCut_6$ under action of Σ_6 .

The subgroup Σ_2 of the full symmetry group $\Sigma_6 \times \Sigma_2$ contracts some pairs of orbits of the group Σ_6 into one orbit of the full group. The result is given in the forth row of Table.

Note that the symmetry groups of Cut_7 and $OCut_6$ have 36 and 37 orbits of facets, respectively.

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