LOGARITHMIC DISTANCES IN GRAPHS

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Abstract: *The walk distances in graphs are defined as the result of appropriate transformations of the* $\sum_{k=0}^{\infty}(tA)^k$ *proximity measures, where* A *is the weighted adjacency matrix of a graph and* t *is a sufficiently small positive parameter. The walk distances are graph-geodetic; moreover, they converge to the shortest path distance and to the so-called long walk distance as the parameter* t *approaches its limiting values. Furthermore, the logarithmic forest distances which are known to generalize the resistance distance and the shortest path distance are a specific subclass of walk distances. On the other hand, the long walk distance is equal to the resistance distance in a transformed graph.*

Keywords: *graph distances, walk distances, logarithmic forest distances, transitional measure, Laplacian matrix, resistance distance, network*

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MSC*: 05C12, 05C50, 15B48*

The classical distances for graph vertices are the well-known shortest path distance, the resistance distance, which is proportional to the commute time distance, and the square root version of the resistance distance. The latter two distances were first studied by Gerald Subak-Sharpe in the 60s. Recently, a need for a wider variety of graph distances has been strongly felt (see [Deza and Deza, 2009; von Luxburg, Radl, and Hein, 2011; Tang, 2010; Estrada, 2011] among many others).

Recall the well-known fact that the shortest path distance and the resistance distance coincide on each tree. In particular, for every path, the resistance distance between every two adjacent vertices is one, as well as the shortest path distance. However, in some applications two central adjacent vertices in a path may be considered as being *closer* to each other than two peripheral adjacent vertices are as there are more walks (of length 3, 5, etc.) connecting two central vertices. Such a "gravitational" property holds for the forest distances we studied since 1995. In some other applications, a terminal vertex in a path can be considered as being closer to its neighbor than two central adjacent vertices are. For example, if someone has a single friend, then this friendship is often stronger than that between persons having more friends. This heuristic is supported by the logarithmic forest distances [Chebotarev, 2011].

In [Chebotarev, 2011a], a general framework was proposed for constructing graph-geodetic metrics (a distance $d(i,j)$ for graph vertices is graph-geodetic whenever $d(i,j)+d(j,k) = d(i,k)$ if and only if every path connecting i and k visits j). Namely, it has been shown that if a matrix $S = (s_{ij})$ produces a strictly positive transitional measure on a graph G (i.e., $s_{ij}\,s_{jk} \le s_{ik}\,s_{jj}$ for all vertices i, j, and k, while $s_{ij}\,s_{jk} = s_{ik}\,s_{jj}$ if and only if every path from i to k visits j), then the logarithmic transformation $h_{ij} = \ln s_{ij}$ and the inverse covariance mapping $d_{ij} = h_{ii} + h_{jj} - h_{ij} - h_{ji}$ convert S into the matrix of a graph-geodetic distance. In the case of digraphs, five transitional measures were found in [Chebotarev, 2011a], namely, the "connection reliability", the "path accessibility" with a sufficiently small parameter, the "walk accessibility", and two versions of the "forest accessibility".

Earlier, the inverse covariance mapping has been applied to the matrices of walk weights $\sum_{k=0}^{\infty}(tA)^k$, where A is the adjacency matrix of a graph. This leads to distances whenever the positive parameter t is sufficiently small. However, these distances are not graph-geodetic and some of their properties are quite exotic.

In the present paper, we study the graph-geodetic *walk distances*, which involves the logarithmic transformation. The walk distances are expressed in terms of commute cycles and via block matrix operations. Two limiting cases

of walk distances are investigated: the short walk distance coincides with the classical shortest path distance, while the long walk distance is original. Furthermore, modified walk distances (the "e-walk distances") are considered which generalize the classical *weighted* shortest path distance. It is shown that adding "balancing loops" converts the logarithmic forest distances into a subclass of walk distances. This implies, in particular, that the resistance distance is also a limiting walk distance. Finally, several graph metrics are compared on simple examples.

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