FUZZY SETS: MATH, APPLIED MATH, HEURISTICS? PROBLEMS AND INTERPRETATIONS

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***Abstract**: Number of Disciplines and Theories changed their status from status of Natural Science discipline to Mathematics. The Theory of Probability is the classical example of that kind. The main privilege of the new Math status is the conception of Math truth, which distinguishes Math from other theories. Some disciplines used in Applications pretended to be Math not really being it. It is entirely true of Fuzzy Subsets Theory with its pretension to be Math and to be exclusive tools in uncertainty handling. Fundamental pretensions of classical Fuzzy subset theory including pretension to be math as well as some gaps in the theory are discussed in the article.*

Statistical interpretation of membership functions is proposed. It is proved that such interpretation takes place for practically all supporters with minimal constraints on it. Namely, a supporter must be a space with a measure. Proposed interpretation explains modification of classical fuzzy objects to fill the gaps. It is then possible to talk about observations of fuzzy subset within the conception of modification and to extend likelihood method to the new area. Fuzzy likelihood equation is adduced as an example of new possibilities within the proposed approach. One more interpretation for the Fuzzy subset theory is proposed for a discussion: multiset theory.

***Keywords**: f Uncertainty, Plural model of uncertainty, Fuzzy subsets Theory, statistical interpretation of the membership function, modification of Fuzzy subsets, Fuzzy likelihood equation, Multiset theory.*

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Introduction

Initially, the concept of fuzziness was intended to be the object of the proposed article. But later it became clear that the extent of the issue ought to be wider. First of all, the expansion must include a discussion about the role of fuzziness within the concept of uncertainty. What is the "uncertainty" itself? Is it mathematics? If not, where ought one to look for the origin of the concept? How does uncertainty sort with fuzziness? Which one of these two is primary? There are more pertinent questions related to the place of Math and Applied Math in definitions and applications of uncertainty and fuzziness as well as to the role of Heuristics in Applied researches. Uncertainty, surely, is the first in the discussion about priority within the mentioned pair. Pospelov, for instance, and his school [Поспелов, 2001] share this opinion. They consider fuzziness to be the means for handling uncertainty but not vice versa. As to Math, Applied Math, it is worth mentioning in connection with the Fuzzy subset theory (FzTh) coming into the world thanks to Lotfi Zadeh [Zadeh, 1965] (see also [Kaufmann, 1982]), that it was proclaimed to be mathematical panacea for uncertainty modeling.

Mathematics

There is a principal consideration determining the relations between Math and Empiric experience.

As to Mathematics itself, Wikipedia, for example, [Wikipedia, Math] states the following: "Mathematics is the study of quantity, structure, space, and change. Mathematicians seek out patterns, formulate new conjectures, and establish truth by rigorous deduction from appropriately chosen axioms and definitions." Thus, specific objects (Math structures) and conception of Math truth (rigorous deduction) are the essence of Mathematics.

As to the Math structures (see for example [Донченко, 2009]). When saying "math structure" we ought to understand it as a set plus "bonds" or "relations" between the elements of the set. Correspondent "bonds" or "relations" in Math are specified by: 1) Math relations (for example "$\leq$" in R^1); 2) functions; 3) operations (for example "+", "$\cdot$" in R^1); 4) collections of subsets (for example, collection of open subsets, collection of closed subsets or collection of neighbours in R^1); 5) combinations of the previous four. All Math structures were initially established for sets of numbers of different kinds: integers, real, complex. Then they were extended on abstract sets. Therefore, we now have, for example, a structure of metric space (an abstract set plus real valued non negative function of two arguments with certain properties), structure of a group, including an affine one (an abstract set plus binary operation with certain properties); structure of linear space (an abstract set with the structure of affine group plus product operations for each real number); structure of Euclidean and Hilbert space (structure of linear space plus non-negative real-valued function of two arguments: scalar product); topological space (an abstract set plus an appropriate collection of its subsets, with possibility to define the limit), measurable space (an abstract set plus a collection of its subspace, named by σ-algebra), linear topological space and so on. More detailed structure may be considered within the base structure: linear subspace or hyper plane within linear or Euclidean space, a subgroup within the group and so on.

Math truth

The fundamental concept of Math truth is the concept of deducibility. It means that the status of truth (proven statement) is given to the statement which is terminal in the specially constructed sequence of statements called its proof. It is a peculiarity in sequence constructing that the next in it is produced by the previous one by special admissible rules (deduction rules) from initial, admissible statements (axioms and premises of a theorem). As a rule, corresponding admissible statements have a form of an equation with formulas on both its sides. So, each subsequent statement in the sequence-proof of the terminal statement is produced by the previous member of the sequence (equation) by changing a part of the formula on its left or right side to another one: from another side of equations-axioms or equations premises. The specification of restrictions on admissible statements and the deduction rules are the object of math logic.

Applied Mathematics

The main aim of Applied Math (AppMa) is description of a real object by Math. This means that the object under consideration as "a structure" is represented by means of Math structure, i. e. main parts of the object under modeling and principal bonds, ties, relation between them are represented be means of Math structurization. It is a necessary condition for math modeling to have apt interpretation for correspondent Math objects and objects under observation. Such interpretations, for example for a function and its derivatives, are correspondingly path and speed. Integration and differentiation are the means to represent the relation between the speed and path. Likewise, frequencies of that group or those groups of results in a sequence of observations are interpreted as probabilities and vice versa. Surely, those interpretations can be applied under certain restrictions. So, we cannot investigate discrete systems by means of differential equations or apply probabilistic method out of fulfilling low-frequencies stability. The main aim of Math description, Math modeling of a real object, is to take advantage of establishing true statements for an apt Math object (target statement) which represents the real object: for its Math model, with following interpretation of correspondent statements. So, if the model of the real object is an equation, the target math statement is the statement on its decision and following interpretation of the decision for the real object. So, the following three-step procedure is the essence of AppMa. 1. Math decrypting of an object on the base of available knowledge about the real object under consideration and with the help of an apt interpretation. 2. Establishing math truth for apt (target) statements within the Math model. 3) Interpretation of the target math statements for the real object. The first and the last steps are impossible without interpretation. The availability of interpretation is principal for applied math. Thus, interpretation plus math rigorous truth are the essence of applied math. So, for example, numerology is not Math and AppMa because it does not appeal to target (Math truth) statements.

Heuristics

There are some more means of research which use Math at this stage of investigation but they do not follow the three-step procedure of AppMa. They may be designated by “intellectual calculus” as some investigators do it, but it is reasonable, to my mind, to use an old apt word “heuristic”. Indeed, [by Wikipedia: heuristics] “heuristic or heuristics (from the Greek "Εὑρίσκω" for "find" or "discover") refers to experience-based techniques for problem solving, learning, and discovery. Heuristic methods are used to speed up the process of finding a good enough solution, where an exhaustive search is impractical. Examples of this method include using a "rule of thumb", an educated guess, an intuitive judgment, or common sense”. Taking advantage of opportunity, we mention here the authors of the “heuristic” from Polya [Polya,1945] through A.Newell&J.C. Shaw& H.A.Simon[Newell& Shaw& Simon,1962] to D. Kahneman [Wikipedia: Kahneman].

Uncertainty

It is common for investigators to say or use the expression “modeling under uncertainties”. It is also generally recognized that the theory of probability is the classical means for uncertainty handling when such uncertainty is shown as randomness. Determination of randomness appeals to the notion of experiment (observation, trail, test, sometimes – stochastic experiment). Thus, to understand what randomness is, it is necessary to look into the conception of “experiment”.

Experiment

As analysis of numerous sources on Theory of Probability and Math Statistics [Донченко 2009] shows, the notion of experiment is associated with what is described as *conditions* (conditions of experiment) under which a phenomenon is investigated, and with what occurs under the conditions, described as *the results of experiment.*

Therefore, as in [Донченко 2009], it is proposed to consider “experiment” as the pair *(c, y)*: *c*- conditions of experiment (observation, trail, and test), y – result of experiment. Henceforth, Y_c for the fixed condition *c* will denote the set of all possible events that may occur in the experiment under conditions $c \in C$. Generally speaking Y_c is not singleton.

It is reasonable to mark out in a condition c variation, controlled, part x: $x \in R^p$ as a rule and part f, which is invariable by default in a sequence of experiment. Condition c under such approach is denoted by the pair: c=(x, f), $x \in X \subseteq R^p$.

Sequence of Experiments and their registration

If there are *n* experiments, then their registration is the sequence below:

$$(c_i, y_i), c_i \in C, y_i \in Y_{c_i}, i = \overline{1,n}. \qquad (1)$$

Different variants of (1) can be implemented in practice:

$$((x_i, f), y_i), x_i \in X \subseteq R^p, y_i \in Y_{x_i}, i = \overline{1,n}, \quad (2)$$

$$(x_i, y_i), x_i \in X \subseteq R^p, y_i \in Y_{x_i}, i = \overline{1,n}, \quad (3)$$

$$y_i, y_i \in Y_{c_i}, i = \overline{1,n}, \quad (4)$$

$$y_i, y_i \in Y, i = \overline{1,n} \quad (5)$$

It is obvious that (5) is equivalent to (1) when all conditions are the same:

$$c_i \equiv c, i = \overline{1,n}.$$

But if otherwise, then

$$Y = \bigcup_{i=1}^{n} Y_{c_i} \neq \text{Singleton}$$

Randomness as a classic example of uncertainty

Randomness in designation introduced above means, firstly, that the results of the experiment are not determined by conditions $c \in C$ definitely, i. e.

$$Y_c \neq \text{Singleton}, c \in C. \quad (6)$$

And, secondly, that the observations satisfy low-frequency stability. This means: 1) in a sequence of experiment with fixed conditions c, frequency of each collection of possible results from Y_c turn to some limit value; 2) the limit value does not depend on the sequence of observations and characterizes the phenomenon under consideration. In the Theory of Probability Y_c is called Space of elementary events and is denoted by Ω. Corresponding experiment is often called stochastic experiment.

Plural model of uncertainty

As randomness is a special kind of uncertainty, so the definition of uncertainty one ought to look for is in the conception of experiment. Then, the natural definition of uncertainty coincides with the first part of randomness and is described by equity (5). Thus, uncertainty is defined on the basis of experiment and classifies certain relation between conditions of experiment *c* and its corresponding results *y*. This relation is stated in (5). We will name such conception of uncertainty Plural Model of Uncertainty (PluMoU).

Mathematical means for uncertainty handling

There are comparatively few math tools for uncertainty handling. Having no possibility to discuss the issue in detail, we would like to at least mention that these are: 1)Theory of Probability; 2)Inverse Problem; 3) MaxMin method; 4) Hough Transform; 5) Multisets Theory; 6) Fuzzy Theory; 7) combination of 1)-6) issues. The last point needs additional explanation in order to embed FzTh in PluMoU. Such embedding becomes feasible on the basis of two possible interpretation of FzTh: within Theory of Probability and Multisets theory.

Fuzzy Theory and statistical interpretation of membership function

Fuzzy set $\underline{A}$, subset to be more precise (Kaufmann, 1982), as the object in mathematics is nothing more than a graphic image of real valued function μ on an abstract crisp (usual) set *E* (henceforth - supporter of the Fuzzy subset). There is additional constraint on the value of this function, named membership function in Fuzzy theory: its values are bounded by the segment $[0,1]$:

$$\mu: E \to [0,1] \ , \underline{A} = \{(e, \mu_{\underline{A}}(e)) : e \in E)\} .$$

There are no objections. The definition is perfect but trivial. There are great many functions in mathematics, there are great many graphics and there are no pretensions of the Fuzzy theory.

Limitations of Fuzzy Theory

As it was mentioned above, there are several Math tools for uncertainty handling. All of them are well-grounded Math. So, FzTh is not exclusive in pretension to uncertainty handling. Also, the attention was drawn earlier to the importance of interpretation in Applied Math unlike in fundamental. As to FzTh, the lack of objective interpretation is a rather acute problem. The absence of its own set theory as well as a Fuzzy logic is the problem awaiting solutions. There are some steps relating to logic (see, for example, [Hajek, 1998F], [Hajek, 1998]), but the problem of interpretation in this case must also be solved. The importance of apt interpretation may be clearly demonstrated on history of modal logic.

There is no such a thing as axiomatic set theory in FzTh even in naive, Kantor's sense. Particularly, such axiom of paramount importance known as abstraction [Stoll, 1960] or separation principle [Kuratovski, Mostowski,1967] is out of consideration. Implementation of a variant of this axiom in FzTh would help to overcome the "object" problem. Indeed, as is well known, the axiom under consideration establishes correspondence between classical (crisp) subsets and properties of the elements of a universal set – namely, predicates on the universal crisp set. So, a classical predicate has its object of characterization: the correspondent set, determined by abstraction axiom. In FzTh changing binary predicates by membership functions there is no defining of other elements in the pair (predicate, set). As a consequence, the object of fuzzy characterization is lost. Incidentally, Multiset theory (see below) with its technique could help solve the problem.

It is interesting that in obvious examples of membership functions out of the FzTh such objects are intrinsic to the definition of the correspondent objects. Namely, such examples are generalized variants of logit- and probit (GeLoPr) – regressions, transition matrix for Markov's chains and Bayesian nets are the examples mentioned.

Natural examples of membership function: generalized variants of logit and probit regression

As to these examples, GeLoPr describes dependence of the frequencies (probabilities) of a certain event A on real valued vector under certain parameterization:

$$P\{A \mid H_x\} = G(\beta^T \begin{pmatrix} 1 \\ x \end{pmatrix}),$$

$$\beta \in R^{n-1}, \beta^T = (\beta_{0,} \ldots, \beta_{n-1,}), x \in R^n ,$$

Where G – distribution function F(z), $x \in R^1$ or correspondent tail: 1-F(z) for the scalar distribution.

In this example, GeLoPr $\mu(x)=P\{A \mid H_x\}, x \in R^{n-1} = E$ as a function of $x \in R^{n-1}$ is a membership function in classical FzTh, which corresponds to the certain object intrinsic to the theory: event A. It is important to remember that the event A, mentioned above, describes presence of a certain property in observation $(x, y), y \in \{0,1\}$. The value 1 for y means fulfilling and 0 - not fulfilling the property in the observation.

Natural examples of membership function: Markov chain

A transition matrix for the Markov's chain $(\xi_n, n \in \mathbb{N})$, with stated set $\wp = \{S_1,...,S_M(...)\}$ is the $M \times M$ matrix $P = (p_{ij})$ of conditional probabilities:

$$p_{ij} = P\{\xi_{n+1} = S_j \mid \xi_n = S_i\}, i, j = \overline{1,M}$$

Each column with number $j = \overline{1,M}$ of the matrix defines membership function $\mu_j, j = \overline{1,M}$ on $E = \wp$:

$$\mu_j(S_i) = p_{ij} = P\{\xi_{n+1} = S_j \mid \xi_n = S_i\}, j = \overline{1,M}, \quad (6)$$
$$S_i \in \wp = E$$

In each of the M membership functions $\mu_j(S), S \in \wp = E, j = \overline{1,M}$, there are intrinsic objects of fuzzy characterization. Namely, these are correspondingly: $\{\xi_{n+1} = S_j\}, j = \overline{1,M}$.

It is interesting that it is natural to consider (6) to be a "full system" of membership functions: a collection of functions $\mu_j, j = \overline{1,M}$ on E, for which, for any $e \in E$, there is:

$$\sum_{j=1}^{M} \mu_j(e) = 1, e \in E$$

Natural examples of membership function: Bayesian nets

Any Bayesian net is, in the essence, a weighted directed graph associated with probabilistic objects. But, when in a classic probabilistic graph the weights are prescribed to the edges with one and the same head-nodes, in Bayesian – to the one with the same tail-nodes. Thus, the collection of probabilities is associated with each node: the probabilities which weigh the nodes of predecessors. So, correspondent probabilities (conditional by its nature) define a membership function.

Probabilistic Interpretation of membership function

This subsection deals with probabilistic interpretation of the classical variant of FzTh (Donchenko, 1998, 3). Two variants of a supporter E are considered below: discrete and non-discrete. Discrete case is the one which fully illustrates the situation. Namely, each membership function of a fuzzy subset is represented by a system of conditional probabilities of certain events, relatively complete collections of the sets $H_e, e \in E$. Saying "complete collection" we consider the collection $H_e, e \in E$ to be the partition of the space of elementary events Ω for a basic probability space.

Probabilistic Interpretation of membership function: discrete supporter

The main point of the subsection is represented by theorem 1 [Donchenko, 1998, 3].

Theorem 1. For any classical Fuzzy Set $(E, \mu_{\underline{A}}(e))$ with discrete support E, there exist such discrete probability space

$$(\Omega, B_\Omega, P),$$

event

$$A \in B_\Omega$$

and complete collection of events

$$H_e : H_e \in B_\Omega, e \in E$$

within this probability space, such that membership function $\mu_A(e)$ is represented by the system of conditional probabilities in the following form:

$$\mu_{\underline{A}}(e) = P(A \mid H_e), e \in E. \qquad (7)$$

Theorem 2. For any complete collection of Fuzzy subsets $(E, \mu_{A_i}(e)), i = \overline{1,n}$, with one and the same supporter E there exist:

discrete probability space (Ω, B_Ω, P);

collection of evens $A_i \in B_\Omega, i = \overline{1,n}$;

complete collection of events $H_e : H_e \in B_\Omega, e \in E$ within the probability space (Ω, B_Ω, P),

such, that all of the membership functions $\mu_{A_i}(e), e \in E, i = \overline{1,n}$ are simultaneously represented by the systems of conditional probabilities in the following way:

$$\mu_{\underline{A}_i}(e) = P(A_i \mid H_e), e \in E, i = \overline{1,n}.$$

Probabilistic Interpretation on membership function: non discrete supporter

The issue in the previous subsection may be extended noticeably to non-discrete case if the supporter E possesses certain structure, namely, if it is space with a measure [Donchenko, 1998, 3].

Theorem 3. Given that:

$(E, \Im, m)$ - is space with a measure;

$(E, \mu_{\underline{A_i}}(e)), i = \overline{1, n}$, is complete Fuzzy subsets collection with the same supporter E;

all of the membership functions $\mu^{(A_i)}(e), i = \overline{1, n}$, are $\Im, £,$ - measurable (£ – Borel σ-algebra on R^1),

then, there exist: probability space (Ω, B_Ω, P),

ξ- discrete random S_P – valued random variable on (Ω, B_Ω, P), where S_P is any n -element set with elements, say, $S_i, i = \overline{1, n}$;

η random E – valued random variable on (Ω, B_Ω, P) such that for any $i = \overline{1, n}$

$\mu^{(A_i)}(e) = P\{\xi = S_i \mid \eta = e\}$,

where

$P\{\xi = S_i \mid \eta\}$

– conditional distribution of random variable(r.v) ξ respectively r.v. η.

The conditional distribution is regular: for any $e \in E$ $P\{B \mid \eta = e\}$ there is a probability B respectively.

Remark on the proof. The proof is the result of extended ideas of the previous theorems, but it embodies an application of another technique: the technique of conditional distribution. The proof, however, being technically complicated is omitted here.

Remark 1. There are obvious objects of uncertainty characterization within the theorems 1-3.

Modified definition of fuzzy sets

Straight reference to the object or uncertainly described property may be, in the author's opinion, a way to solve the problem of constructing an analog of the separation principle. This reference ought to be reflected evidently in the definition of the membership function:

$$\mu^{(T)}(e), e \in E,$$

where T – correspondent property (predicate) on certain set U. The last is the set of “uncertain characterization”. It may coincide with E. So $\mu^{\{T\}}(e)$ would be “uncertain characterization” of property T or corresponding crisp subset $P_T \subseteq U$. The last transition is possible owing to the separation principle for crisp sets. Two membership functions $\mu^{\{T_1\}}(e)$ and $\mu^{\{T_2\}}(e)$ with $T_1 \neq T_2$ would specify two different Fuzzy sets, even if they are equal as the function of $e, e \square E$.

Definition. The pair

$(E, \mu^{(T)}(e))$

or

$(E, \mu^{(P_T)}(e))$

is called modified Fuzzy subset (MoF) with E as a supporter, which uncertainly describes crisp T on U (or correspondent crisp subset, $P_T \subseteq U$, where U- the “universal” crisp set of “uncertain characterization”), if:

E – is the abstract crisp set, which is referenced to as a supporter;

T - is a crisp predicate on U correspondingly, P_T - crisp subset of U, which corresponds to T;

$\mu^{(T)}(e) \in [0,1]$ – function of two arguments: $e, e \in E$ and T from the set of all crisp predicates on universal crisp U.

The function

$$\mu^{(T)}(e), e \in E$$

just as in classical theory of Fuzzy sets, will be referenced to as membership function, with note that it uncertainly characterizes property T (or correspondent subset P_T).

Remark 2. Obviously, statistical interpretation of the theorems 1-3 is applicable to MoF.

Observations of the Modified Fuzzy Sets

The modification of Fuzzy set definition introduced earlier in the paper imparts objectivity to Fuzzy sets and allows for a discussion about observations of modified Fuzzy sets (Donchenko, 2004). It's a very important ontological aspect of mathematical modeling using Fuzzy sets. The observation of modified Fuzzy sets is the pair $(e, T(e)) - e, e \in E$ – element from the supporter and T(e) is the predicate value on this element. Namely, e is the element, displayed in observation and $T(e)$ is the fixed information about fulfilling the property T in the observation, specified by $e \in E$. It is just in such a way the observations are interpreted in the logit- and probit – regressions and in its generalizations.

So, the observation sample is $(e_i, t_i), t_i = T(e_i), i = \overline{1, n}$ and we can talk about independent observation within statistical interpretation.

Likelihood method for modified fuzzy sets

Statistical interpretation of a membership function grants the possibility to talk about an extension of statistical MLM for estimating fuzzy parameter just as it happens in the regressions mentioned above.

Indeed, let

$$\mu^{(T)}(e), e \in E$$

-MoF with membership function from parametric collection of membership functions

$$\mu^{(T)}(e) = \mu(e, \beta), \beta \in R^p .$$

Let $(e_i, t_i), i = \overline{1, n}$ independent observation of MoF. We determine "Fuzzy Likelihood function" $FL(\beta)$) by the relation

$$FL(\beta) = \prod_{i=1}^{n} \mu^{t_i}(e_i, \beta)(1 - \mu(e_i, \beta))^{1-t_i} .$$

Correspondingly, we denote by

$$fl(\beta) = \ln FL(\beta) = \sum_{i=1}^{n} t_i \ln \mu(e_i, \beta) +$$

$$+ \sum_{i=1}^{n} (1 - t_i) \ln(1 - \mu(e_i, \beta))$$

- logarithmic "Fuzzy Likelihood function".

Just as it is in statistical likelihood estimation

$$\hat{\mu}^{(T)}(e) = \mu(e, \hat{\beta}),$$

where

$$\hat{\beta} = \arg\max_{\beta \in R^p} FL(\beta).$$

Just as in Statistics if $\mu^{(T)}(e) = \mu(e, \beta)$, necessary condition is the

$$\frac{\partial FL(\beta)}{\partial \beta} = 0$$

or

$$\frac{\partial fl(\beta)}{\partial \beta} = 0.$$

The last equation is equivalent to the first one under additional restriction that the set of zeroes of $\mu(e, \beta), \beta \in R^p$, respectively $\beta \in R^p$ is the same for all $e \in E$.

It is reasonable to refer to the equations of necessary conditions as “fuzzy likelihood equations”.

Theorem 4. Under all necessary restrictions “fuzzy likelihood equations” are of the following form:

$$\sum_{i=1}^{n} \frac{t_i - \mu(e_i, \beta)}{\mu(e_i, \beta)((1 - \mu(e_i, \beta)))} \frac{\partial \mu(e_i, \beta)}{\partial \beta_j} = 0,$$

$$j = \overline{1, p}, \beta = \begin{pmatrix} \beta_1 \\ \cdots \\ \beta_p \end{pmatrix} \in R^p.$$

Expert estimating can also be used by combining LSM and MLM.

Multisets Theory

Multisets (see, for example, reviews: [Blizard, 1989], [Буй, Богатирьова 2010]) are the Math's answer to the necessity of describing sets with elements which may “repeat”. Thus, originally the concept of multiset implements the idea of repetition rep(u) for elements u from subset D of a certain universal set U. Which are the sets D and U, and correspondingly rep(u), depends on peculiarities of applied problem. So, for example, D can be a set of answers for this or that call in the Internet, answers and rep(u) – number of repetition for each record. There is a natural way of implementing the idea of repetition: to provide each $u \in D$ with a number or repetition $n_u : n_u \in \{1, 2, \ldots, n, \ldots\} \equiv N^+$.

So, we get the first variant for determining multiset 1. We will call the multiset the set of pairs $\bigcap_{u \in D} \{(u, n_u)\}, n_u \in N^+, u \in D \subseteq U$ for any subset D of certain universal set U. Then, D will be the base of the multiset and n_u -multiplicity or repetition factor. The terms will be used in all variants of multiset definitions below in an evident way. Multiset with base D will be denoted by $D^{(ms)}$.

Thus, multiset $D^{(ms)}$ is the usual set D with “comments” n_u to its elements.

2. Within the frame of the second definition, multiset for any subset D of a certain universal set U is the transformation $\alpha : D \to N^+$, defined for any $u \in D$ (see, for example, [Петровский, 2002], [Редько, 2001]) . Equivalence of the first and second determination is evident: $\alpha(u) = n_u, u \in U$. One ought to remark that in the second variant the relation function substitutes the set.

3. Third variant: $D^{(ms)}$ for $D \subseteq U$ is the pair $D^{(ms)} \equiv (D, \alpha) : \forall D \subseteq U, \forall \alpha : D \to N^+$, α is defined on all elements of D. Thus, in this variant multiset is the pair: set D –"comment" α .

When necessary, we will refer to the components of the multiset-pair $D_{ms} = (D, \alpha)$ in an evident way, correspondingly, by D_α , and α_D as well as by $D_{D^{(ms)}}, \alpha_{D^{(ms)}} : D = D_{D^{(ms)}}, \alpha_{D^{(ms)}}(u) = \alpha(u), u \in U$.

Natural set terminology is applied for multisets: for standard operations("$\cup$","$\cap$") and for standard relation: "$\subseteq$". We will denote them for multisets correspondingly "$\cup_{ms}$","$\cap_{ms}$"" $\subseteq_{ms}$ ".

We will define them

$$\forall D_1^{(ms)} = (D_1, \alpha_1), D_2^{(ms)} = (D_2, \alpha_2) : D_i \subseteq U, i = 1,2$$

by the relations, correspondingly:

1. $D_1^{(ms)} \subseteq D_2^{(ms)} \Leftrightarrow (D_1 \subseteq D_2 \ \& \ \alpha_1 \le \alpha_2)$

2. $D_1^{(ms)} \cup_{ms} D_2^{(ms)} \equiv (D_1 \cup D_2, \max(\alpha_1, \alpha_2))$

3. $D_1^{(ms)}{}_{ms} \cap D_2^{(ms)} \equiv (D_1 \cap D_2, \min(\alpha_1, \alpha_2))$

As to operation "$\bar{\ }$", it is necessary to "cut" N^+ to $N_M^+ = \{1, 2, \ldots, M\}$ leaving the rest of the determinations unchangeable. Then $\overline{D^{(ms)}} = \overline{(D, \alpha)}$ is determined by the relation

$$\overline{D^{(ms)}} = (D, M - \alpha)$$

Characteristic function $\chi_{D^{(ms)}}(u)$ (see, for example, [Buy, Bogatyreva, 2010]) is convenient in multiset handling. It is determined by the relation

$$\chi_{D^{(ms)}}(u) = \begin{cases} \alpha(u), u \in D \\ 0, \quad u \notin D \end{cases}$$

Namely, characteristic function is an extension of repetition factor or multiplicity on the universal set U.

The role of characteristic functions in multiset theory is fixed by the equivalency in determination of set operations and order described by the following relations:

1. $(D_1^{(ms)} \subseteq D_2^{(ms)}) \Leftrightarrow (\chi_{D_1^{(ms)}} \le \chi_{D_2^{(ms)}})$

2. $\chi_{D_1^{(ms)} \cup D_2^{(ms)}} = \max(\chi_{D_1^{(ms)}}, \chi_{D_2^{(ms)}})$,

3. $\chi_{D_1^{(ms)} \cap D_2^{(ms)}} = \min(\chi_{D_1^{(ms)}}, \chi_{D_2^{(ms)}})$.

Multisets Theory and Fuzziness

It is evident that in multiset theory, repetition factor is the “absolute” variant of membership function. By this we mean absolute and relative frequency. Even more, in the variant of using N_M^+ we get pure membership function by dividing repetition factor α by M. However, there are essential differences between these two theories: all membership functions in FTh are referenced to one and the same E (U in the designations of the multiset theory) and are referenced to the particular $D \subseteq U$ in multiset theory. Simple substitution: subsets D instead of one and the same universal set in Fth solves the problem of the object characterization: D is the object. All other limitations of the Fth are also immediately solved: 1) own set theory with corresponding set operations and order; 2) own logic: commonly used mathematical logic; 3) interpretation of ($\alpha(u)$ as rep(u)); 4) abstraction axiom: for each $D \subseteq U$ there are many possible correspondent α : any of them.

Conclusion

General approach to describing uncertainty was expounded in the paper within conception of plurality in understanding uncertainty. The uncertainty is the quality of interaction between the researcher and phenomenon within an observation (experiment, trial, and test). Obviously, some formalization for the “observation” is proposed and discussed in the text. The proposed concept of uncertainty makes it possible for all to use math means for uncertainty handling. It is entirely true of Fuzzy approach after proving principal theorems on statistical interpretation of membership function. Some limitations of Fuzzy Theory were discussed and several examples and directions for overcoming them were demonstrated. Namely, these were modifications proposed for the membership function and Multiset Theory.

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ROUGH SET METHODS IN ANALYSIS OF CHRONOLOGICALLY ARRANGED DATA

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***Abstract**: The paper presents results of efforts of increasing predicting events accuracy by increasing a set of attributes describing the present moment by information included in past data. There are described two experiments verifying such an approach. The experiments were carried on by the use of the RSES system, which is based on the rough sets theory. The data analyzed in the first experiment, concerning the weather, were reported at the meteorological station in Jasionka near Rzeszów from 1 April 2004 to 30 september 2005. The second experiment deals with exchange rates based on the money.pl news bulletin data (http://www.money.pl).*

***Keywords**: rough sets, prediction, temporal data.*