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(editors)

New Trends in Information Technologies

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CHAIN SPLIT OF PARTIALLY ORDERED SET OF K-SUBSETS

Hasmik Sahakyan, Levon Aslanyan

***Abstract**: An application oriented class of partially ordered sets is considered. Let $P(n,k)$ denotes the set of all k-tuples with strictly increasing elements from the set $N=\{1,2,\cdots,n\}$ and $1 \le k \le n$. Some properties of $P(n,k)$ is studied in terms of partially ordered sets. An algorithm that constructs a set of non intersecting increasing chains that cover all elements of $P(n,3)$ is brought. The number of these chains is the minimal possible: it equals to the width of $P(n,3)$, i.e. the largest cardinality of an antichain. Analogous to the Hansel's well known algorithm for identification of monotone Boolean functions, the chains constructed for $P(n,3)$ can be used for identification of monotone functions defined on $P(n,3)$.*

***Keywords**: partially ordered sets, chain split.*

***ACM Classification Keywords**: G.2.1 Discrete mathematics: Combinatorics*

Introduction

An application oriented class of partially ordered sets is considered. Let $P(n,k)$ denotes the set of all strictly increasing k-tuples of elements that are from the set $N=\{1,2,\cdots,n\}$, and for some $k, 1 \le k \le n$. Properties of $P(n,k)$ and consequently of $P(n,3)$ is studied. An algorithm that constructs a set of non intersecting increasing chains that cover all elements of $P(n,3)$ is brought. Analogous to the Hansel's well known algorithm for identification of monotone Boolean functions, based on partitioning of the set of vertices of the cube into non intersecting chains, - the chains, constructed for $P(n,3)$ can be used for identification of monotone functions given in $P(n,3)$. The study of $P(n,3)$ is also motivated by its tight relation with the 3-hypergraphs.

Partially Ordered Sets

This section brings introduction to the partially ordered sets ([ST, 2008], [E, 1997]).

Definition 1. A partially ordered set (or poset) is an ordered pair $(P,\le)$, consisting of a set P and a relation $\le$ on P satisfying the following three properties:

(1) for all $x \in P$, $x \le x$ (reflexivity).

(2) for all $x,y \in P$, if $x \le y$ and $y \le x$, then $x = y$ (anti-symmetry).

(3) for all $x,y,z \in P$, if $x \le y$ and $y \le z$, then $x \le z$ (transitivity).

The notation $x < y$ is used when both $x \le y$ and $x \ne y$.

Definition 2. An element x of a poset P is minimal if there is no element $y \in P$ s.t. $y < x$. Similarly, x is maximal if there is no element $z \in P$ s.t. $x < z$.

Two elements x and y in the poset P are comparable if $x \le y$ or $y \le x$; otherwise x and y are incomparable.

Definition 3. A *chain* in a poset $(P,\le)$ is a subset C of P which is totally ordered in P. An *antichain* is a set A of pairwise incomparable elements.

The *height* of a poset is the largest cardinality of a chain, and its *width* is the largest cardinality of an antichain. We denote the height and width of $(P,\leq)$ by $h(P)$ and $w(P)$. In a finite poset $(P,\leq)$, a chain C and an antichain A have at most one element in common.

Theorem 1 (*Dilworth's Theorem) Let $(P,\leq)$ be a finite poset. Then there is a partition of P into $w(P)$ chains.*

Let x and y be distinct elements of a poset $(P,\leq)$. We say that y *covers* x if $x<y$ but no element z satisfies relation $x<z<y$. The *Hasse diagram* of a poset $(P,\leq)$ is the directed graph whose vertex set is P and whose arcs are the covering pairs (x,y) in the poset. We usually draw the Hasse diagram of a finite poset in the plane in such a way that, if x is covered by y, then the point representing y is higher than the point representing x. Then no arrows are required in the drawing, since the directions of the arrows are implicit

While Dilworth's theorem uses transitive comparisons in splitting $(P,\leq)$ into the chains, our interest below concerns the chains consisting of pairwise covering vertices, that is chains in the Hasse diagram. A general postulation on existence of such chain splits on posets is not known. It is not hard to compose a simple poset P that can't be split into $w(P)$ increasing chains of covering vertices. Such decompositions are valid for a number of well known particular cases of posets such as the unit cube, the structure of unit cube subcubes by inclusion, etc. The poset that we investigate in this regard is the special order of k-subsets of a finite set, - the equivalent structure of the k-th layer of a unit cube.

k-subsets

Let $P(n,k)$ denotes the set of all k-tuples with strictly increasing elements from the set $N=\{1,2,\cdots,n\}$, and for some $k, 1\leq k\leq n$. $(i_1,\cdots,i_k)\in P(n,k)$ iff $1\leq i_1<i_2<\cdots<i_k\leq n$. The number of elements of $P(n,k)$ is C_n^k. For two elements, $(i_1,\cdots,i_k)$ and $(j_1,\cdots,j_k)$ we define $<$ relation as follows: $(i_1,\cdots,i_k)<(j_1,\cdots,j_k)$ if and only of $i_1<j_1,\cdots,i_k<j_k$. Then $P(n,k)$ becomes a poset with minimal and maximal elements $(1,2,\cdots,k)$ and $(n-k+1,\cdots,n)$ respectively. We define the weight of $(i_1,\cdots,i_k)$ as the sum of its coordinates, $i_1+\cdots+i_k$. Now let us form the Hasse diagram of $P(n,k)$. The lowest layer of the diagram consists of the unique vertex $(1,2,\cdots,k)$. Then the i-th layer consists of all elements of $P(n,k)$ that cover some elements of the $(i-1)$-th layer. The highest layer contains the vertex $(n-k+1,\cdots,n)$. So the overall diagram consists of $k\cdot(n-k)+1$ layers: we number them from 0 to $k\cdot(n-k)$.

All vertices of the i-th layer have equal weights which is $i+(1+\cdots+k)$. We introduce a notion of middle layer or layers, which is the ($k\cdot(n-k)/2$)-th layer for even k, or odd k and odd n; and the ($k\cdot(n-k)\pm 1$)-th layers for odd k and even n.

Each layer of $P(n,k)$ consists of pairwise incomparable elements, that is, it composes an antichain. According to the Dilworth theorem there is a partition of $P(n,k)$ into $w(P(n,k))$ chains. We will prove that $w(P(n,k))$ is achieved among the vertex sets of layers of $P(n,k)$. Our attention is restricted to the case $k=3$ in regard to the framework of describing 3-hypergraph degree sequences [S, 2009], where 3 is the minimal number to check the complexity of algorithms for the hypergraph degree sequence problem [B, 1986]. $w(P(n,3))$ is found, and increasing chains consisting of covering vertices, are constructed for $P(n,3)$.

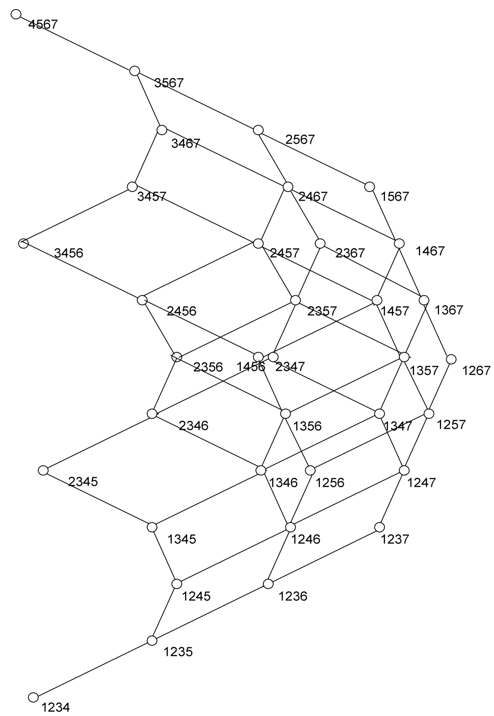


Figure 1. Hasse diagram of $P(7,4)$.

$P(n,3)$

In this section we give formula for calculating layer cardinalities of $P(n,3)$ and study some properties of $P(n,3)$ that will be used for determining the largest cardinality and to construct splitting to the chains.

Formula

Let L_l denotes the layer of $P(n,3)$ containing elements with the weight equal to l :

$L_l = \{(i_1, i_2, i_3) / \ i_1 + i_2 + i_3 = l, 1 \le i_1 < i_2 < i_3 \le n\}$. Calculation of $|L_l|$ below is done by determining the range of feasible values for each coordinate i_j.

It is easy to check that the minimal feasible value for i_1 is $\max(1, l - 2n + 1)$ and the maximal value equals $\lfloor l/3 \rfloor - 1$.

For a given feasible i_1 the minimal feasible value of i_2 is $\max(i_1 + 1, l - i_1 - n)$ and the maximal is $\lceil (l - i_1)/2 \rceil - 1$.

For given i_1 and i_2, - i_3 is unique.

Resuming the above reasoning, we bring the formula of $|L_l|$:

$$|L_l| = \sum_{i_1 = \max(1, l-2n+1)}^{\lfloor l/3 \rfloor - 1} \left(\lceil (l - i_1)/2 \rceil - \max(i_1 + 1, l - i_1 - n) \right).$$

For determining the layer of greatest cardinality which we intend, the formula given is improper, and we study further properties of $P(n,3)$.

Symmetry

The elements of $P(n,3)$ located on j-th layer have weight $j+6$. Middle layer of $P(n,3)$ is at $L_{mid}=\frac{3\cdot(n-3)}{2}$ for odd n and there are two middle layers $L_{mid+}=\frac{3\cdot(n-3)+1}{2}$ and $L_{mid-}=\frac{3\cdot(n-3)-1}{2}$ for even n. $P(n,3)$ is symmetric in respect to its middle layer (layers). If j-th layer contains an element (i_1,i_2,i_3) for some j, then its "opposite" element that we define as $(n+1-i_3,n+1-i_2,n+1-i_1)$ is located on the $(3\cdot(n-3)-j)$-th layer. We denote by $\widehat{P}(n,3)$ and $\breve{P}(n,3)$ the parts of $P(n,3)$ above and below the middle layers respectively.

Partitioning

The structure of $P(n,3)$ naturally partitioned into 3 parts, denote them by $P^1(n,3)$, $P^2(n,3)$ and $P^3(n,3)$. $P^1(n,3)$ and $P^3(n,3)$ consists of the first and last $n-3$ layers of $P(n,3)$ respectively, and $P^2(n,3)$ consists of the remaining $n-3+1$ layers.

Consider a layer i from the part $P^1(n,3)$. It is simple to indicate one specific vertex $(1,2,i+3)$ on this layer, which is used in forthcoming considerations. Symmetrically, $P^3(n,3)$ contains opposite to i layer $3(n-3)-i$ and the vertex $(n-i-2,n-1,n)$ on it. Our main attention is to the middle part $P^2(n,3)$. We count the layer widths of $P^2(n,3)$ from layers 1 to $n-3+1$, and indicate the vertex $(1,i+1,n)$ for the layer $i+1$. Obviously middle layer or layers belong to $P^2(n,3)$.

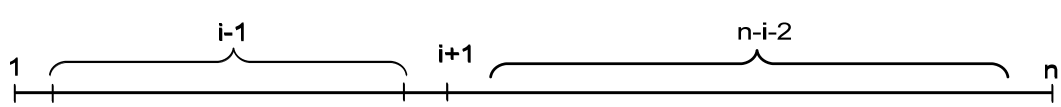
Quantities *in* $P^1(n,3)$ and $P^2(n,3)$

Elements of i-th layer of $P^1(n,3)$ can be generated starting from $(1,2,i+3)$: a group of elements is generated by increasing the second coordinate 2 and decreasing the third one $i+3$ simultaneously. Then consider $(2,3,i+1)$ and generate elements by increasing 3 and decreasing $i+1$. In general consider $(1+j,2+j,i+3-2j)$ while $1+j\le\left\lfloor\frac{1+2+i+3}{3}\right\rfloor-1$, that is $j\le\left\lfloor\frac{i}{3}\right\rfloor$, and generate elements by increasing the second coordinate and decreasing the third. It follows that the number of such elements increases with i, and therefore $P^1(n,3)$ has maximal number of elements on its last $(n-3-1)$-th layer. Completely analogous is the situation for $P^3(n,3)$.

Quantities *in* $P^2(n,3)$

Now consider the middle zone $P^2(n,3)$.

We construct the vertices (a,b,c) of a particular layer $i+1$ in this area. This layer as we know contains the vertex $\alpha_{i+1}=(1,i+1,n)$ which will be the origin of our constructions.



Construction will be done by several groups. The groups are defined as sets of vertices that have first coordinate fixed, for example, for α_{i+1} it is 1. For the first coordinate a we denote the corresponding group by G_{i+1}^{a}. In G_{i+1}^{a} let $b_{\min}$ be the smallest possible value for the second coordinate, denoted it by b. The third coordinate which we denote by c, is determined in a unique way by a and b given. Let $c_{\max}$ is the greatest possible value for the third coordinate for fixed a (it is determined by $b_{\min}$).

For a given a define *group operation* for generating all elements of the group. First compute $b_{\min}$ and $c_{\max}$ for a, then the group operation increases $b_{\min}$ by one (shifts the position to the right) and decreases $c_{\max}$ by one (shifts the position to the left). Evidently this group consists of all elements of layer $i+1$, having a its first coordinate. All the groups by different first coordinates are non intersecting. Thus $\bigcup_{a} G_{i+1}^{a}$ represents the layer $i+1$ of $P^{2}(n,3)$. Moreover, it is easy to calculate the group sizes when we know $b_{\min}$ and $c_{\max}$: it simply equals $1+\lfloor (c_{\max}-b_{\min}-1)/2 \rfloor$ or the same $\lfloor (c_{\max}-b_{\min}+1)/2 \rfloor$.

Further we do two types of actions – compute the group sizes for all feasible a's, and compute and compare the groups of neighbor layers $i+1$ and i. Last action intends to determine the layer of maximum size in $P^{2}(n,3)$.

We start from α_{i+1}. Then increase a of α_{i+1} by one. To keep the vertex in the same layer we decrease the second coordinate b by one, the third, c remains the same, - currently it is n. Repeating this operation while new b is greater than new a, we get series of vertices of layer $i+1$. These vertices have the property that $b=b_{\min}$ and $c=c_{\max}=n$ for the a fixed. Determine the values of a and b at the end of these series. 2 cases are possible: a) a and b meet at $a=(i-1)/2$ and $b_{\min}=(i-1)/2+1$ for even $i-1$. b) a and b meet at the positions $(i-2)/2$ and $(i-2)/2+2$ when $i-1$ is odd. In case b) the value $(i-2)/2+1$ between a and b is not used. Then increasing a and decreasing c by one we get the element $((i-2)/2+1,(i-2)/2+2,n-1)$, which generates an additional group. Denote this group by G_{i+1}^{*}. All groups constructed at this stage are called $G1_{i+1}$ groups.

Now a still can be increased while it reaches the great possible value for a, that is: $a=\lfloor (1+i+1+n-3) \rfloor/3$. Continue increasing a. At this stage increasing a causes increasing also b. Increase a and b by one (this is the smallest b, that is $b_{\min}$) and decrease c by two (this is $c_{\max}$). Repeating these operation, which ends at a triple (a,b,c) with $a=\lfloor (1+i+1+n-3) \rfloor/3$, we get new groups of elements, that we call $G2_{i+1}$ groups.

Below the table represents two series of groups, first for layer $i+1$ and second for i. Consider the case when $i-1$ is odd.

Layer $i+1$	$G1_{i+1}^{1}$	$G1_{i+1}^{2}$	$\cdots$	$G1_{i+1}^{(i-2)/2}$	G_{i+1}^{*}	$G2_{i+1}^{(i-2)/2+2}$	$\cdots$
Layer i	$G1_{i}^{1}$	$G1_{i}^{2}$	$\cdots$	$G1_{i}^{(i-2)/2}$	$G2_{i}^{(i-2)/2+2}$	$G2_{i}^{(i-2)/2+3}$	$\cdots$

An important notion is that all groups $G1_{i}^{1}, G1_{i}^{2}$, ..., $G1_{i}^{(i-2)/2-1}$ are congruent to groups $G1_{i+1}^{2}, G1_{i+1}^{3}$, ..., $G1_{i+1}^{(i-2)/2}$ correspondingly, their sizes are equal and they might be eliminated in comparisons of layers i and $i+1$. The case when $i+1$ is even is similar to this one.

Groups and their sizes

$G1_{i+1}^{1+j}$ The group consists of elements where $a = 1+j$, $b = i+1-j$ and $c = n$: $(1+j, i+1-j, n)$. Possible values for j are $0,1,\cdots,(i-1)/2$ for even $i-1$ and $0,1,\cdots,(i-2)/2$ for odd $i-1$. The subgroup of each j is generated by the group operation. So the subgroup of j contains:

$$1+\left\lfloor \frac{n-(i+1-j)-1}{2} \right\rfloor = \left\lfloor \frac{n-i+j}{2} \right\rfloor \text{ elements.}$$

The last subgroup starts with the element $(1+(i-1)/2, 2+(i-1)/2, n)$, or the same $((i+1)/2, 1+(i+1)/2, n)$ for even $i-1$ and $((i+1)/2, 2+(i+1)/2, n)$ for odd $i-1$.

So we get $\sum_{j=0}^{\frac{i-1}{2}} \left\lfloor \frac{n-i+j}{2} \right\rfloor$ elements for even $i-1$ and $\sum_{j=0}^{\frac{i-2}{2}} \left\lfloor \frac{n-i+j}{2} \right\rfloor$ for odd $i-1$.

G_{i+1}^{*} This group exists only for odd $i-1$. It starts with the element $(2+(i-2)/2, 3+(i-2)/2, n-1)$, or the same $(1+i/2, 2+i/2, n-1)$, and generates $\left\lfloor \frac{n-1-(2+i/2)-1}{2} \right\rfloor$ elements. So $\left|G_{i+1}^{*}\right| = \left\lfloor \frac{n-4-i/2}{2} \right\rfloor$.

$G1_{i+1}$ is the union of all these sets: $G1_{i+1} = \left(\bigcup_j G1_{i+1}^{1+j} \right) \bigcup G_{i+1}^{*}$.

$G2_{i+1}$ group described above consists of elements, where $a = 1+(i-1)/2+j = (i+1)/2+j$, $b = 2+(i-1)/2+j = 1+(i+1)/2+j$ and $c = n-2j$, where possible values for j are $1,\cdots,$ while $(i+1)/2+j \le \left\lfloor \frac{i+n-1}{3} \right\rfloor$, that is, $j \le \left\lfloor \frac{i+n-1}{3} \right\rfloor - \frac{i+1}{2}$, - for even $i-1$ and $i/2+j \le \left\lfloor \frac{i+n-1}{3} \right\rfloor$, $j \le \left\lfloor \frac{i+n-1}{3} \right\rfloor - \frac{i}{2}$, for odd $i-1$. The subgroup of each j is generated by the group operation. So the subgroup of j contains $1+\left\lfloor \frac{n-2j-(1+(i+1)/2+j)-1}{2} \right\rfloor$ elements.

$\left|G2^{i+1}\right| = \sum_{j=1}^{\left\lfloor \frac{i+n-1}{3} \right\rfloor - \frac{i+1}{2}} \left\lfloor \frac{n-2j-((i+1)/2+j)}{2} \right\rfloor$ for even $i-1$. For odd $i-1$

$$\left|G2^{i+1}\right| = \sum_{j=1}^{\left\lfloor \frac{i+n-1}{3} \right\rfloor - \frac{i}{2}} \left\lfloor \frac{n-2j-(i/2+j)}{2} \right\rfloor$$

All the above reasoning prove the following theorem:

Theorem: $G1_{i+1}$ and $G2_{i+1}$ are non intersecting groups that cover the $(i+1)$-th layer of $P^2(n,3)$.

Our next goal is to find the areas of increasing cardinalities among the neighbor layers. Compose $G1$ and $G2$ groups for the i-th layer of $P^2(n,3)$. We will consider the case of even $i-1$ only. The case of odd $i-1$ can be done in an analogous way.

$G1_i^{1+j}$ is the group of elements where $a = 1 + j$, $b = i - j$ and the third is $c = n$: $(1 + j, i - j, n)$, where possible values for j are $0, 1, \cdots, (i - 1)/2 - 1$. The subgroup of each j is generated by the group operation. So the subgroup of j contains:

$1 + \left\lfloor \frac{n - (i - j) - 1}{2} \right\rfloor = \left\lfloor \frac{n - i + j + 1}{2} \right\rfloor$ elements. The last subgroup starts with the element $((i - 1)/2, (i - 1)/2 + 2, n)$. So we get $\sum_{j=0}^{\frac{i-1}{2}-1} \left\lfloor \frac{n - i + j + 1}{2} \right\rfloor$ elements.

Here we have an additional group.

$G1_i^*$ starts with the element $((i - 1)/2 + 1, (i - 1)/2 + 2, n - 1) = ((i + 1)/2, 1 + (i + 1)/2, n - 1)$, which generates $\left|G1_i^*\right| = \left\lfloor \frac{n - 1 - (i + 1)/2}{2} \right\rfloor$ elements by the group operation. Then $G1_i$ is the union of these subgroups: $G1_i = \left(\bigcup_j G1_i^{1+j} \right) \bigcup G_i^*$.

$G2_i$ This group contains elements where $a = 1 + (i - 1)/2 + j = (i + 1)/2 + j$, $b = 1 + (i + 1)/2 + j$ and $c = n - 1 - 2j$, where possible values for j are $1, \cdots,$ while $\frac{i + 1}{2} + j \le \left\lfloor \frac{n + i - 2}{3} \right\rfloor$, that is $j \le \left\lfloor \frac{n + i - 2}{3} \right\rfloor - \frac{i + 1}{2}$. Then the subgroup of each j is generated by the group operation. So the subgroup of j contains 1+$\left\lfloor \frac{n - 1 - 2j - (1 + (i + 1)/2 + j) - 1}{2} \right\rfloor$ elements.

$$\left|G2_i\right| = \sum_{j=1}^{\left\lfloor \frac{n+i-2}{3} \right\rfloor - \frac{i+1}{2}} \left\lfloor \frac{n - 2j - (1 + (i + 1)/2 + j)}{2} \right\rfloor$$

Calculate the differences: $\left|G1_{i+1}\right|$ - $\left|G1_i\right|$ and $\left|G2^{i+1}\right| - \left|G2^i\right|$.

$$\left|G1_{i+1}\right| - \left|G1_i\right| = \sum_{j=0}^{\frac{i-1}{2}} \left\lfloor \frac{n - i + j}{2} \right\rfloor - \sum_{j=0}^{\frac{i-1}{2}-1} \left\lfloor \frac{n - i + j + 1}{2} \right\rfloor - \left\lfloor \frac{n - 1 - (i + 1)/2}{2} \right\rfloor = \left\lfloor \frac{n - i}{2} \right\rfloor - \left\lfloor \frac{n - 1 - (i + 1)/2}{2} \right\rfloor$$

$$\left|G2^{i+1}\right| - \left|G2^i\right| = \sum_{j=1}^{\left\lfloor \frac{n+i-1}{3} \right\rfloor - \frac{i+1}{2}} \left\lfloor \frac{n - 2j - ((i + 1)/2 + j)}{2} \right\rfloor - \sum_{j=1}^{\left\lfloor \frac{n+i-2}{3} \right\rfloor - \frac{i+1}{2}} \left\lfloor \frac{n - 2j - 1 - ((i + 1)/2 + j)}{2} \right\rfloor$$

Consider cases:

a) 3 is divisor of $n + i - 1$, it follows that $\left\lfloor \frac{n + i - 2}{3} \right\rfloor = \left\lfloor \frac{n + i - 1}{3} \right\rfloor - 1$

b) $(n+i-1)/3$, 1 remainder, it follows that $\left\lfloor \frac{n+i-2}{3} \right\rfloor = \left\lfloor \frac{n+i-1}{3} \right\rfloor$

c) $(n+i-1)/3$, 2 remainder, it follows that $\left\lfloor \frac{n+i-2}{3} \right\rfloor = \left\lfloor \frac{n+i-1}{3} \right\rfloor$

Consider b) or c)

$$|G2_{i+1}| - |G2_i| = \sum_{j=1}^{\left\lfloor \frac{n+i-1}{3} \right\rfloor - \frac{i+1}{2}} \left(\left\lfloor \frac{n-2j-((i+1)/2+j)}{2} \right\rfloor - \left\lfloor \frac{n-2j-1-((i+1)/2+j)}{2} \right\rfloor \right) =$$

$$\sum_{j=1}^{\left\lfloor \frac{n+i-1}{3} \right\rfloor - \frac{i+1}{2}} \left(\left\lfloor \frac{n-(i+1)/2-3j}{2} \right\rfloor - \left\lfloor \frac{n-(i+1)/2-1-3j}{2} \right\rfloor \right)$$

1) $n-(i+1)/2$ is even, then it follows that $n-(i+1)/2-3j$ is even for even j and is odd for odd j.

1a) $n-(i+1)/2$ is even and j is even, and then it follows that

$$\left\lfloor \frac{n-(i+1)/2-3j}{2} \right\rfloor = \left\lfloor \frac{n-(i+1)/2-1-3j}{2} \right\rfloor + 1.$$

1b) $n-(i+1)/2$ is even and j is odd, then it follows that $\left\lfloor \frac{n-(i+1)/2-3j}{2} \right\rfloor = \left\lfloor \frac{n-(i+1)/2-1-3j}{2} \right\rfloor$

So in case 1a) $|G2_{i+1}| - |G2_i| = \sum_{\text{even } j}^{\left\lfloor \frac{n+i-1}{3} \right\rfloor - \frac{i+1}{2}} 1$, approximately the half of the upper index.

2) $n-(i+1)/2$ is odd, then it follows that $n-(i+1)/2-3j$ is odd for even j and is even for odd j.

2a) $n-(i+1)/2$ is odd and j is even, and then it follows that

$$\left\lfloor \frac{n-(i+1)/2-3j}{2} \right\rfloor = \left\lfloor \frac{n-(i+1)/2-1-3j}{2} \right\rfloor.$$

2b) $n-(i+1)/2$ is odd and j is odd, and then it follows that $\left\lfloor \frac{n-(i+1)/2-3j}{2} \right\rfloor = \left\lfloor \frac{n-(i+1)/2-1-3j}{2} \right\rfloor + 1.$

So in case 2b) $|G2_{i+1}| - |G2_i| = \sum_{\text{odd } j}^{\left\lfloor \frac{n+i-1}{3} \right\rfloor - \frac{i+1}{2}} 1$, approximately the half of the upper index.

Further analysis of all possible cases provides that the $(n+1)/2$-th (for odd n) and $n/2$-th and $(n/2+1)$-th (for even n) layers of $P^2(n,3)$, - serve as layers of the largest cardinality for $P(n,3)$. In both cases these are the middle layers.

Chain Split

In this part our goal is to split $P(n,3)$ into disjoint chains of covering pair sequences. Then each chain must contain exactly one element of the antichain of largest cardinality, and consequently will pass through the middle layer/layers. Due to the symmetry property it is sufficient to have chain constructions only for $\widehat{P}(n,3)$ or $\breve{P}(n,3)$ and then the extended construction is by symmetry.

Notice that the antichain of the largest cardinality contains the element $(1,(n+1)/2,n)$ for odd n and the 2 antichains of the largest cardinality contains $(1,n/2,n)$ and $(1,n/2+1,n)$ respectively, for even n.

Algorithm

1. **Ordering of elements.** Consider lexicographic order of elements on layers;

2. **Constructing chain fragments in $\breve{P}(n,3)$.** Consider a recurrent procedure. First chain starts with the element $(1,2,3)$. Any current chain starts with the smallest unused element of the lowest layer that still contains unused elements and goes up until it reaches the layer L_{mid} for odd n (L_{mid+} for even n). From this point we go up by increasing the third component until it reaches n or the middle layer. If we come in some step across an element which is already used in previous chains, then we go back and increase the second component by one and then continue increasing the third. If also the increase of second component moves the element to the used one, then we go back and increase the first component by one, and continue increasing the second, etc. until the chain reaches the middle layer or finds a deadlock. For an element e from L_{mid} (L_{mid+}) we denote by $C(e)$ the chain reaching this element. The first chain that started at $(1,2,3)$ reaches $(1,(n+1)/2,n)$ for odd n and $(1,(n/2+1),n)$ – for even n.

As an example consider $P(9,3)$. $159, 168, 249, 258, 267, 348, 357, 456$ lists the elements of layer L_{mid}.

Chains in $\breve{P}(9,3)$ constructed by the algorithm are:

$C(159) = \{123, 124, 125, 126, 127, 128, 129, 139, 149, 159\}$,

$C(168) = \{134, 135, 136, 137, 138, 148, 158, 168\}$,

$C(249) = \{234, 235, 236, 237, 238, 239, 249\}$, $C(267) = \{145, 146, 147, 157, 167, 267\}$

$C(258) = \{245, 246, 247, 248, 258\}$, $C(357) = \{156, 256, 257, 357\}$,

$C(348) = \{345, 346, 347, 348\}$, $C(456) = \{356, 456\}$.

3. **Extending chains to the $\widehat{P}(n,3)$.** Complete chains of $P(n,3)$ are constructed in a way of extending the chains of $\breve{P}(n,3)$ into the $\widehat{P}(n,3)$ area. We use the symmetry property of $P(n,3)$ in the following way. For each element $e=(i_1,i_2,i_3)$ of L_{mid}, it is easy to check that its "opposite" to $e=(i_1,i_2,i_3)$- the element $e^{op}=(n+1-i_1,n+1-i_2,n+1-i_3)$ also belongs to L_{mid}, for odd n. When n is even, for each element $e=(i_1,i_2,i_3)$ of $L_{(mid-)}$, its "opposite" element $e^{op}=(n+1-i_1,n+1-i_2,n+1-i_3)$ belongs to $L_{(mid+)}$, and vice versa. For the above example: $159^{op}=159$, $168^{op}=249$, $249^{op}=168$, $267^{op}=348$, $258^{op}=258$, $357^{op}=357$, $348^{op}=267$, $456^{op}=456$.

Then continuation of a chain $C(e)$ of $\breve{P}(n,3)$ into the $\widehat{P}(n,3)$ area considers the chain denoted by $C^{up}(e)$, that consists of all "opposite" elements $C(e^{op})$ of the $C(e)$ taking in inverse order.

$C^{up}(159) = \{149^{op}, 139^{op}, 129^{op}, 128^{op}, 127^{op}, 126^{op}, 125^{op}, 124^{op}, 123^{op}\} =$

$\{169, 179, 189, 289, 389, 489, 589, 689, 789\}$, $C^{up}(168) = \{178, 278, 378, 478, 578, 678\}$,

$C^{up}(249) = \{259, 269, 279, 379, 479, 579, 679\}$, $C^{up}(267) = \{367, 467, 567\}$,

$C^{up}(258) = \{268, 368, 468, 568\}$, $C^{up}(357) = \{358, 458, 459\}$,

$C^{up}(348) = \{349, 359, 369, 469, 569\}$, $C^{up}(456) = \{457\}$.

So we get the chains:

$\{123, 124, 125, 126, 127, 128, 129, 139, 149, 159, 169, 179, 189, 289, 389, 489, 589, 689, 789\}$

$\{134, 135, 136, 137, 138, 148, 158, 168, 178, 278, 378, 478, 578, 678\}$

$\{234, 235, 236, 237, 238, 239, 249, 259, 269, 279, 379, 479, 579, 679\}$

$\{145, 146, 147, 157, 167, 267, 367, 467, 567\}$

$\{245, 246, 247, 248, 258, 268, 368, 468, 568\}$

$\{156, 256, 257, 357, 358, 458, 459\}$

$\{345, 346, 347, 348, 349, 359, 369, 469, 569\}$

$\{356, 456, 457\}$

And the whole construction given by the algorithm is illustrated in the figure 2.

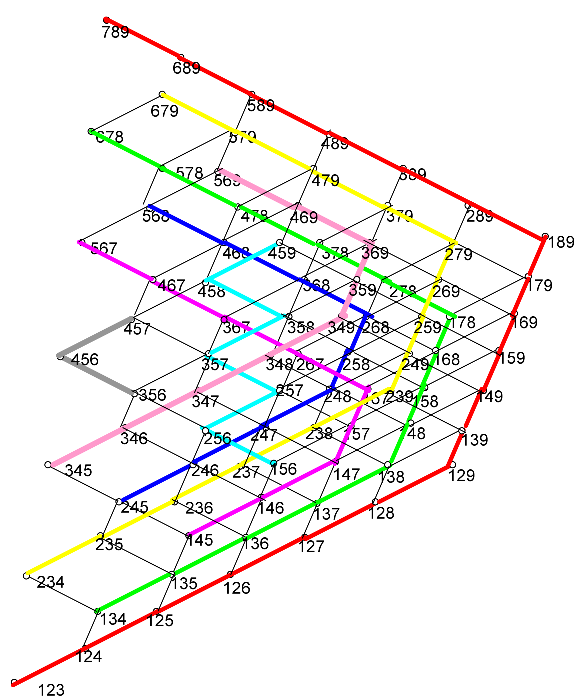


Figure 2

Correctness of the algorithm. First is the claim about the deadlock free of algorithms when growing the chains in $\breve{P}(n,3)$. Then, it is to prove that chains constructed in step 2 cover all the elements of $\breve{P}(n,3)$ symmetrically.

This two steps are done by induction on n taking into account the structure of l-th layer of $P(n,3)$ that is a union of $\leq l$ layers of $P(n-i,2)$, for $i=1,\cdots$

Final comparisons and computation of the chains are by formulas given above.

Conclusion

We constructed non intersecting increasing chains that cover all elements of $P(n,3)$. Theoretical outcome is that in addition to the chains by Dilworth's theorem we prove the existence of chains consisted of covering elements, - as an analogy to the Hansel chains for binary cubes. The practical outcome is the monotone recognition of subsets C_k of elements of k-th layer of the n dimensional unit cube by the use of queries about the involvement of several vertices into the C_k.

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